TWO LANGUAGES OF SCHUBERT CALCULUS: GRASSMANNIANS AND FLAG MANIFOLDS

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ABSTRACT. We focus on the question why flag manifolds generalize the Grassmannian in terms of Schubert calculus. On our way we discuss Schubert cells and Schubert varieties in these two languages. The latter ones form a basis for the cohomology rings, respectively. This leads to structure constants. We introduce Schubert polynomials and Schur polynomials and relate these to cohomology. Finally, we present the translation between Grassmannian and flag manifold Schubert calculus.

1. INTRODUCTION

In this paper, we will give a brief overview of the theory of Schubert calculus in two contexts: On the Grassmannian $Gr_{k,n}$ (i.e. the space of k-planes in $\mathbb{C}^n$) and on the flag manifold Fl_n. In fact, both of these are examples of *partial flag manifolds*, the space $Fl_n^{(d_1,d_2,\ldots,d_m)}$ consisting of partial flags $F_1 \subset F_2 \cdots \subset F_m$ in $\mathbb{C}^n$ where F_i is a subvectorspace of $\mathbb{C}^n$ having dimension d_i. Schubert calculus can be formulated in this more general setting in a fashion which encompasses both the Grassmannian and flag theories. However, for most purposes this generality is not necessary and to some extent it obscures the theory in the Grassmannian case. We will begin by defining the basic geometric objects. We take the definition of the Grassmannian for granted, but will briefly review the definition of the flag manifold.

2. GRASSMANNIANS AND FLAG MANIFOLDS

A (complete) *flag* $F_\bullet$ in $\mathbb{C}^n$ is a strictly ascending chain $\{0\} = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n$ of subspaces of $\mathbb{C}^n$, where F_i is a linear subspace of dimension i. Given a flag $F_\bullet$, if another flag $G_\bullet$ is choosen randomly (we make this precise later) then we expect the intersections of the subspaces in $F_\bullet$ and those in $G_\bullet$ to be as small as possible, i.e. $\dim(F_i \cap G_j)$ should be $\max(i+j-n,0)$. In such a case, $F_\bullet$ and $G_\bullet$ are said to be in *transverse position* or simply transverse. In particular, for any flag $F_\bullet$, if we define the flag $F_\bullet^\perp$ by setting $F_i^\perp$ to be the space perpendicular to F_{n-i}, then $F_\bullet$ and $F_\bullet^\perp$ will be transverse.

The set of all flags can be thought of as the topological space of matrices in $\mathrm{GL}_n(\mathbb{C})$ modulo the relation where two matrices A and B are equivalent if, for all i between 1 and n, the first i columns of A span the same subpace as the first i columns of B. We define the subspace $X_\perp(F_\bullet)$ to be the set of all flags transverse to $F_\bullet$. These sets are open in the previous topology (intuitively, if we have a flag

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$G_\bullet$ transverse to $F_\bullet$ and perturb it slightly then its intersection dimensions with $F_\bullet$, these are $F_i \cap G_j$, will not increase) and in fact give the set of all flags the structure of a differentiable manifold which we will call Fl_n.

There is a natural transisitive action of $\mathrm{GL}_n(\mathbb{C})$ on Fl_n given by $A \cdot F_\bullet = \{0 \subset A \cdot F_1 \subset A \cdot F_2 \subset \cdots \subset A \cdot X\}$ which is smooth in the above topology.

Lemma 1. *The sets $X_\perp(F_\bullet)$ form an atlas on Fl_n of dimension $\binom{n}{2}$.*

Proof. The sets $X_\perp(F_\bullet)$ certainly cover Fl_n since $F_\bullet \in X_\perp(F_\bullet^\perp)$. To prove the isomorphism statement, first let E_i be the subspace of $\mathbb{C}^n$ spanned by the first i vectors of the standard basis of $\mathbb{C}^n$ and consider the *standard flag* $E_\bullet = \{0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = \mathbb{C}^n\}$. We can write down an isomorphism ϕ between $\mathbb{C}^{\binom{n}{2}}$ and$X_\perp(E_\bullet)$ explicitly by

$$(*, *, \dots, *) \mapsto \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & 0 & 1 & * \\ 0 & 1 & * & \vdots \\ 1 & * & \cdots & * \end{bmatrix}.$$

For any flags $F_\bullet$ and $G_\bullet$, we can choose elements $A_F, A_G \in GL_n(\mathbb{C})$ such that $A_F F_\bullet = A_G G_\bullet = E_\bullet$. The desired isomorphisms $A_F \circ \phi : \mathbb{C}^{\binom{n}{2}} \to X_\perp^o(F_\bullet)$ and $A_G \circ \phi : \mathbb{C}^{\binom{n}{2}} \to X_\perp^o(G_\bullet)$ are C^∞ on the preimage of the overlap of two open sets. $\square$

Flag manifolds and Grassmannians are quite different objects but note that for any Fl_n and any $1 \le k \le n$, there is the *forgetful map* $\pi_k : Fl_n \to Gr_{k,n}$ which maps a flag $F_\bullet$ to F_k. This is clearly a smooth, surjective map and will later be quite important for relating the Schubert calculi on these different spaces.

The Plücker embeddings. While the manifold structures of $Gr_{k,n}$ and Fl_n are important, in order to do Schubert calculus, we actually must consider them as *algebraic varieties* by embedding them in a higher dimensional projective space. This is done by the *Plücker embeddings*. In the Grassmannian case, note that if V is a k-dimensional subspace of $\mathbb{C}^n$, then its image $\wedge^k V$ in $\wedge^k \mathbb{C}^n$ will be a line. In particular, if V is spanned by $v_1, \dots, v_k$, then its image will consist of the line spanned by $v_1 \wedge v_2 \ldots \wedge v_k$. This suggests identifying points in $Gr_{k,n}$ with points in $\mathbb{P}(\wedge^k \mathbb{C}^n)$.

More specifically, let M be a $k \times n$ matrix whose rows span V. Since the rank of M is k, there is at least one $k \times k$ minor of M that is non-zero. For any matrix M and any strictly ascending sequence $I = (i_1, i_2, \dots, i_k)$ with $1 \le i_1 < i_k \le n$, let M_I be the minor $\det(M_{p,i_q})_{1 \le p,q \le k}$. If M' is another matrix whose rows span V then there is an element $A \in \mathrm{GL}_n(\mathbb{C})$ such that $M = AM'$ which will imply that $M_I = \det(A) M'_I$ for all I. Together these facts suggest the following definition::

Definition 2. For the Grassmannian $Gr_{k,n}$, consider the projective space $\mathbb{P}(\wedge^k \mathbb{C}^n)$ and denote $e_{i_1} \wedge \cdots \wedge e_{i_k}$ by $X_{i_1,\dots,i_k}$ where $1 \le i_1 < \cdots < i_k \le n$ and the e_i are the vectors in the standard basis of $\mathbb{C}^n$. For every $V \in Gr_{k,n}$, choose a matrix $M(V)$

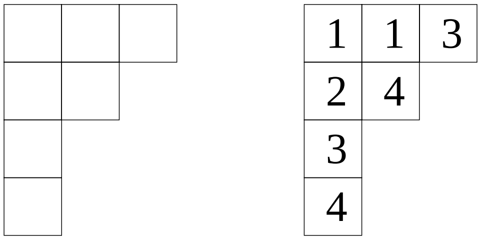

FIGURE 1. Left: Young diagram associated to the partition $\lambda = (3,2,1,1)$. Right: A possible numbering of this Young diagram.

whose rows span V and define a map from $Gr_{k,n}$ to $\mathbb{P}(\wedge^k\mathbb{C}^n)$ by

$$V \mapsto \sum_{I=(i_1,\dots,i_k)} M(V)_I X_{i_1,\dots,i_k}$$

where the sum is over strictly ascending sequences. By the above, the image V under this map is actually independent of the choice of $M(V)$ and so we have a well-defined map from $Gr_{k,n}$ to $\mathbb{P}(\wedge^k\mathbb{C}^n)$ which is called the *Plücker embedding* (that this map is, in fact, an embedding is slightly harder to show, see e.g. **??**).

Now consider Fl_n as the subset of $Gr_{1,n} \times Gr_{2,n} \times \cdots \times Gr_{n-1,n}$ consisting of all $(F_1, F_2, \dots, F_{n-1})$ such that the F_i form a flag. This can then be embedded into $\prod_{k=1}^{n-1}\mathbb{P}(\wedge^k\mathbb{C}^n)$ by the Plücker embeddings for the Grassmannian and the condition of forming a flag cuts out a subvariety known as the *incidence variety*.

3. SCHUBERT CELLS AND SCHUBERT VARIETIES

3.1. **Grassmannians.** As a tool to deal with certain subsets of the Grassmannian we need the notion of Young diagrams and later on also of numberings of Young diagrams. For a more complete introduction to these objects see [3].

Definition 3. Each weakly decreasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ of non-negative integers can be associated to a *Young diagram* consisting of k left-adjusted rows of boxes, row i having λ_i boxes. A *numbering* of the Young diagram λ is an assignment of the integers from 1 to k to the boxes such that each row is weakly increasing and each column is strictly increasing.

For example (with a little bit of abuse of notation) if $\lambda = (3,2,1,1)$, the Young diagram λ is shown on the left of Figure 1 while a numbering of λ is shown on the right.

Definition 4. Let $F_\bullet$ be a flag in $\mathbb{C}^n$ and λ a Young diagram with k rows each containing at most $n-k$ blocks. Then the *(Grassmannian) Schubert cell* $\Omega^o_\lambda(F_\bullet)$ is the set of $V \in Gr_{k,n}$ such that $\dim(V \cap F_j) = i$ when j is between $n+i-\lambda_i$ and $n+i-\lambda_{i+1}$. Equivalently,

$$\Omega^o_\lambda(F_\bullet) = \{V \in Gr_{k,n} : \dim(V \cap F_{n+i-\lambda_i}) = i, \quad 1 \leq i \leq k\}.$$

The Grassmannian Schubert variety $\Omega_\lambda(F_\bullet)$ is the closure of $\Omega^o_\lambda(F_\bullet)$.

Proposition 5. *The Grassmannian Schubert variety* $\Omega_{\lambda}(F_\bullet)$ *is exactly* $\{V \in Gr_{k,n} : dim(V \cap F_{n+i-\lambda_i}) \geq i, \quad 1 \leq i \leq k\}$.

Proof. To see this, suppose V is an element of $\Omega_\lambda(F_\bullet)$ which is not contained in $\Omega^o_\lambda(F_\bullet)$. This means that some of its intersections with members of $F_\bullet$ have higher dimension than allowed. One can then imagine perturbing V to V' which has the appropriate dimensions. Clearly this is possible to achieve while only slightly changing V and so one can find a sequence of elements of $\Omega^o_\lambda(F_\bullet)$ converging to V. □

Note that if λ and μ are two Young diagrams such that $\lambda \subset \mu$ (i.e., $\lambda_i \leq \mu_i$) then the Schubert conditions for a point to lie in $\Omega_\mu(F_\bullet)$ are a subset of those for $\Omega_\lambda(F_\bullet)$ and so we will have $\Omega_\mu(F_\bullet) \subset \Omega_\lambda(F_\bullet)$. In fact, a stronger condition holds:

Lemma 6. *The Schubert variety* $\Omega_\lambda(F_\bullet)$ *is the disjoint union of the Schubert cells* $\Omega^o_\mu(F_\bullet)$ *for all* $\mu \supset \lambda$.

Proof. That $\Omega^o_\mu(F_\bullet) \subset \Omega_\lambda(F_\bullet)$ follows easily from Proposition 5. Conversely, if V satisfies $\dim(V \cap F_{n+i-\lambda_i}) \geq i$ then one can imagine increasing λ_i until the dimension in fact equals i. Doing this for every i would yield a $\mu \supset \lambda$ such that $V \in \Omega^o_\mu(F_\bullet)$. Finally, the fact that the union is disjoint follows immediately from the definition. □

3.2. **Flag Manifolds.** While most pairs of flags are transverse and therefore intersect in a trivial fashion, in general two flags may have a more complicated intersection structure. In general one would want to consider the *dimension table* of $F_\bullet$ and $G_\bullet$, $D = D(F_\bullet, G_\bullet)$ where $D_{ij} = \dim(F_i \cap G_j)$. This table has a very restricted structure, however, and this allows us to encode it in a very useful way. The key features of the matrix are the positions where the dimension increases and by considering the what this set of positions looks like, it can be seen without too much difficulty that we can in fact encode this information in a permutation $w \in S_n$. More specifically, for any two flags $F_\bullet$ and $G_\bullet$ there exists a unique permutation $w \in S_n$ such that

$$\dim(F_i \cap G_j) = \operatorname{rank}(w[i,j]) \quad \text{for} \quad 1 \leq i, j \leq n.$$

Here, $w[i,j]$ denotes the $i \times j$ principal submatrix of the $n \times n$ permutation matrix w which is defined as follows: The entry (i,j) of w is one if $w(i) = n+1-j$ and zero otherwise (note that this is a flipped version of the standard definition of the matrix of a permutation). Flags $F_\bullet$ and $G_\bullet$ are then said to be in *relative position* w. In particular, $F_\bullet$ and $G_\bullet$ being in transverse position corresponds to them having relative position $\mathrm{id} = (1,2,\ldots,n)$. As an example, $(1,4,3,2)$ has matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and flags $F_\bullet = \{0 \subset \operatorname{span}(e_1) \subset \operatorname{span}(e_1,e_2) \subset \operatorname{span}(e_1,e_2,e_3) \subset \mathbb{C}^4\}$ and $G_\bullet = \{0 \subset \operatorname{span}(e_2) \subset \operatorname{span}(e_2,e_3) \subset \operatorname{span}(e_2,e_3,e_4) \subset \mathbb{C}^4\}$ are in relative position $(1,4,3,2)$.

Definition 7. The Schubert cell $X_w^o(F_\bullet)$ in Fl_n is the set of all flags $G_\bullet$ which are in relative position w with respect to $F_\bullet$. Equivalently,

$$X_w^o(F_\bullet) = \{G_\bullet \in Fl_n : \dim(F_i \cap G_j) = \operatorname{rank}(w[i,j]) \quad \text{for} \quad 1 \leq i,j \leq n\}.$$

The *Schubert variety* $X_w(F_\bullet)$ is the closure of $X_w^o(F_\bullet)$ in the flag manifold Fl_n.

In the Grassmannian case, we saw that the partial ordering on Young diagrams corresponded to the inclusion ordering on Schubert cells. For flag manifolds, a similar condition holds except now the Schubert cells are indexed by permutations and the correct partial ordering of the indices is the *Bruhat order*. For a permutation w, the *length* $l(w)$ of w is defined to be the number of inversions in w, i.e. the number of $i < j$ such that $w(i) > w(j)$. Thus the identity permutation has length zero and $(n, n-1, \ldots, 1)$ has length $\binom{n}{2}$. This gives a preorder on S_n which can be strengthened to a partial order by defining $v \leq w$ if there is a sequence of permutations beginning at v and ending at w such that each is the result of multiplying the last by a length-increasing transposition. Note, for example, that $(1,2,\ldots,n) \leq w \leq (n,n-1,\ldots,1)$ for all $w \in S_n$.

Lemma 8. *If $v \leq w$ in the Bruhat order, then $X_v^o(F_\bullet) \subset X_w^o(F_\bullet)$ (and so $X_v(F_\bullet) \subset X_w(F_\bullet)$) for all $F_\bullet$. Morever, we in fact have that $X_w(F_\bullet) = \bigcup_{v \geq w} X_v^o(F_\bullet)$ where the union is disjoint.*

Proof. For essentially the same reasons as in the Grassmannian case, taking the closure of $X_w^o(F_\bullet)$ corresponds to simply changing the equality in the definition to $\dim(F_i \cap G_j) >= \operatorname{rank}(w[i,j])$. It turns out that $v \geq w$ in the Bruhat order is equivalent to $\operatorname{rank}(v[i,j]) \geq \operatorname{rank}(w[i,j])$ which can be verified for considering only the case where v is minimally greater (i.e. is the result of multiplying w with a length increasing permutation as above) than w (see **??** where this is in fact taken as the definition of the Bruhat order, though one must adjust for his defining the Schubert cells in a order-reversing way than we have here). ☐

As the name suggests, these Schubert varieties are in fact subvarieties of their ambient space. The dimension of $X_w(F_\bullet)$ is given by $\binom{n}{2} - l(w)$. Note that with $\lambda_0 = (n-k)^k = (n-k,\ldots,n-k$ and $w_0 = (n,n-1,\ldots,1)$, we have that $|\lambda_0| = k(n-k)$ and $l(w_0) = \binom{n}{2}$ and so $\Omega_{\lambda_0}(F_\bullet)$ and $X_{w_0}(F_\bullet)$ both have dimension zero. In particular, if the k-plane H is in the Schubert variety $\Omega_{\lambda_0}(F_\bullet)$ then we have that $\dim(H \cap F_k) \geq k$ which implies that $H = F_k$. Thus the Schubert variety $\Omega_{(n-k)^k}(F_\bullet)$ consists of a single point. Similarly in the flag manifold, we have that if $G_\bullet$ is in $X_{w_o}(F_\bullet)$ then in particular $\dim(G_i \cap F_i) = i$ and so $G_\bullet = F_\bullet$. Hence, also $|X_{w_o}(F_\bullet)| = 1$.

4. THE CLASS OF A SUBVARIETY

We now attempt to give a brief description of how one may assign a cohomology class to a subvariety X of a nonsingular algebraic variety V. If X is also nonsingular then it is in fact an orientable manifold and thus has a fundamental class $[X]$ which then can be considered as an element of $H^*(V)$ by sending $[X]$ to $[V] \smile i_*([X])$. The difficulty arises when the *singular locus* of X, i.e. the set of points where X is singular, is non-empty. However even in this case, the nonsingular locus of X will have less than full complex dimension and then one can essentially ignore it on the level of homology. Intuitively, one can envision this as giving X the structure of a simplicial complex in a way such that the singular locus of X forms a subcomplex of lower complex dimension and then one can define the fundamental class of X using only the top dimensional simplices in the complex. A more rigorous definition uses a type of Borel-Moore homology.

This now allows us to assign a cohomology class (called a *Schubert cycle*) for every Schubert variety. As it stands, this cycle will depend on the variety's index (either a Young diagram or a permutation) and the reference flag. However, it turns out that the class of a Schubert variety is independent of the reference flag. We will prove this for the flag manifold and the same proof will work (mutatus mutandis) for the Grassmannian as well.

Lemma 9. *For any w and any two flags $F_\bullet$ and $G_\bullet$, $[X_w(F_\bullet)] = [X_w(G_\bullet)]$.*

Proof. For any $A \in \mathrm{GL}_n(\mathbb{C})$, define the map $i_A : X_w(F_\bullet) \to Fl_n$ to be $G_\bullet \mapsto A \cdot G_\bullet$ and so in particular i_{id} is simply the inclusion of $X_w(F_\bullet)$ into Fl_n. If B is such that $F_\bullet = B \cdot G_\bullet$ then we will have by definition that $i_{B*}([X_w(F_\bullet)]) = [X_w(G_\bullet)]$ since $i_B(X_w(F_\bullet)) = X_w(G_\bullet)$. However since $\mathrm{GL}_n(\mathbb{C})$ is connected, there exists a path from $\mathrm{id} \in \mathrm{GL}_n(\mathbb{C})$ to B which will then induce a homotopy between i_{id} and i_B. Since homotopic maps induce identical maps on homology, we will then have that $[X_w(G_\bullet)] = i_{B*}([X_w(F_\bullet)]) = i_{\mathrm{id}*}([X_w(F_\bullet)]) = [X_w(F_\bullet)]$. $\square$

Thus when referring to a Schubert cycle, we can (and will) omit mention of the reference flag used to define it.

5. THE BASIS THEOREM AND STRUCTURE CONSTANTS

The key fact of Schubert calculus is that the Schubert cycles actually additively generate the cohomology ring. To prove this we will need a technical lemma concerning the Borel-Moore homology of algebraic varieties (for a proof, see Appendix B of [3]).

Lemma 10. *Given a filtration $0 = V_m \subset V_{m-1} \subset \cdots \subset V_0 = V$ of a variety V by algebraic subsets V_i such that $V_i \backslash V_{i+1}$ is a disjoint union of algebraic varieties $U_{i,j}$ all of which are isomorphic to some affine space over $\mathbb{C}$, the cohomology ring $H^*(V)$ is additively generated by the classes of the closures of the $U_{i,j}$.*

Theorem 11 (Schubert Basis Theorem)**.** *If X is either the Grassmannian or the flag manifold, then the Schubert cycles in X additively generate its cohomology ring.*

Proof. We will prove the statement in the Grassmannian case, at which point it follows for the flag manifold with a virtually identical argument. Letting $F_\bullet$ be any flag, we define a filtration of $Gr_{k,n}$ by letting V_i be the union of $\Omega_\lambda(F_\bullet)$ over all λ with $|\lambda| \geq i$. We clearly have that $V_{i+1} \subset V_i$ and, since $\Omega_{0^k}(F_\bullet) = Gr_{k,n}$, $V_0 = Gr_{k,n}$ and so this is in fact a filtration. To see that this filtration satisifies the condition of the above lemma, we note that since $\Omega_\lambda(F_\bullet) = \bigcup_{\lambda \subset \mu} \Omega^o_\mu(F_\bullet)$ so that we also have $V_i = \bigcup_{|\lambda| \geq i} \Omega^o_\lambda(F_\bullet)$ and thus $V_i \backslash V_{i+1} = \bigcup_{|\lambda|=i} \Omega^o_\lambda(F_\bullet)$. Since every Schubert cell is isomorphic to an affine space, the above lemma implies that $H^*(Gr_{k,n})$ is generated by the classes of the closures of the Schubert cells $\Omega^o_\lambda(F_\bullet)$, i.e. by the Schubert cycles. □

Note that if we have an additive basis X_i for the cohomology ring of a space then for any X_i and X_j we will have that $X_i \smile X_j = \sum_k c^k_{ij} X_k$ for some constants c^k_{ij}. These constants will be called the *structure constants* of the cohomology ring.

6. SCHUBERT POLYNOMIALS

To get a more explicit way to do Schubert calculus in the the flag manifold, we want to understand the cohomology ring of this space. It will turn out that the Schubert varieties in Fl_n are in one to one correspondence with the Schubert polynomials which we will define in the section. Using Schur polynomials (which will be defined later) we can state the embedding $H^*(Gr_{k,n}) \to H^*(Fl_n)$ explicitly, making the statement *Flag manifolds are a generalization of the Grassmannian"* precise.

For $i \in \{1, \ldots, n-1\}$ let $s_i \in S_n$ denote the transposition interchanging i and $i+1$. The s_i's generate the symmetric group S_n. Further, S_n acts on $R_n = \mathbb{Z}[x_1, \ldots, x_n]$ by permuting the variables. For a polynomial $f \in R_n$ the polynomial $f - s_i(f)$ is anti-symmetric in x_i and x_{i+1} meaning that if the variables x_i and x_{i+1} are interchanged the sign of $f - s_i(f)$ switches. This implies that $x_i - x_{i+1}$ divides $f - s_i(f)$. So the following is well defined.

Definition 12. Define the $\mathbb{Z}$-linear *differential operator* ∂_i on the polynomial ring R_n by

$$\partial_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}} \qquad \text{for} \quad 1 \leq i \leq n-1.$$

This operator has various properties. For example, $\partial_i(f) = 0$ if f is symmetric in x_i and x_{i+1}. Furthermore, $\partial_i(f)$ is symmetric in x_i and x_{i+1}. Hence $\partial_i(\partial_i(f)) = 0$ for every $f \in R_n$.

Definition 13. Given a permutation $w = s_{a_1} s_{a_2} \cdots s_{a_p}$ where $p = l(w)$, then $\partial_{a_1} \partial_{a_2} \cdots \partial_{a_p}$ is independent of the representation choosen (see Sections 10.3 and 10.4 in [3]. This allows us to define the *Schubert polynomial* $\mathfrak{S}_w$ for every permutation $w \in S_n$ by

$$\mathfrak{S}_w = \partial_{w^{-1}w_0}(x_1^{n-1} x_2^{n-2} \cdots x_{n-1}).$$

Notice that the degree of $\mathfrak{S}_w$ is $l(w)$ since each application of ∂_i reduces the degree by 1.

Computation of Schubert polynomials is not as bad as the definition might suggest. Before we compute an example we collect rules which might help for the computation.

Lemma 14. *a) For any i, $\partial_i(\mathfrak{S}_w) = \mathfrak{S}_{w \cdot s_i}$ if $w(i) > w(i+1)$, and $\partial_i(\mathfrak{S}_w) = 0$ otherwise.*

b) We have $\mathfrak{S}_{w_0} = x_1^{n-1}x_2^{n-2}\cdots x_{n-2}^2x_{n-1}$.
c) For each i *we have* $\mathfrak{S}_{s_i} = x_1 + x_2 + \cdots + x_i$.

Proof. Statement a) is Proposition 10.5a [3], b) follows from the definition. Using a), we conclude $\partial_i(\mathfrak{S}_{s_i}) = 1$ and $\partial_j(\mathfrak{S}_{s_i}) = 0$ if $i \neq j$. This shows c). □

The lemma gives rise to an algorithm to compute Schubert polynomials. We illustrate it by example.

Example 15. Given the permutation $w = (35142)$, we want to compute $\mathfrak{S}_{35142}$. We start out with $\mathfrak{S}_{54321} = x_1^4x_2^3x_3^2x_4$. The idea is to iteratively interchange adjacent numbers if they do not agree with the order in w. For example, we could start by interchanging position **4** and 5. (Interchanging 2 and 3 would also work.) We get by Lemma 14 a)

$$\mathfrak{S}_{54312} = \partial_{\mathbf{4}}(S_{54321}) = x_1^4x_2^3x_3^2.$$

Next, we interchange position **2** and 3 (now the only possibility) and end up with

$$\mathfrak{S}_{53412} = \partial_{\mathbf{2}}(S_{54312}) = x_1^4x_2^2x_3^2.$$

Now interchange position **3** and 4 (or 1 and 2):

$$\mathfrak{S}_{53142} = \partial_{\mathbf{3}}(S_{53412}) = x_1^4x_2^2x_3 + x_1^4x_2^2x_4.$$

Interchange position **1** and 2:

$$\mathfrak{S}_{35142} = \partial_{\mathbf{1}}(\mathfrak{S}_{53142}) = x_1^3x_2^2x_3 + x_1^3x_2^2x_4 + x_1^2x_2^3x_3 + x_1^2x_2^3x_4.$$

Since $l(35142) = 6$ we see that $\mathfrak{S}_{35142}$ indeed has the right degree.

We note that $\{\mathfrak{S}_w : w \in S_n\}$ is a $\mathbb{Z}$-basis for $\{x_1^{i_1}\cdots x_{n-1}^{i_{n-1}} : i_j \leq n-j\}$ which is transversal to S, the subring of symmetric functions in R_n. Hence the Schubert polynomials are explicit polynomial representatives of an integral basis of the quotient $H_n = R_n/S$. It will turn out that $H_n \cong H^*(F\,l_n)$ as rings where the isomorphism is given by $\mathfrak{S}_w \leftrightarrow [X_w]$.

7. SCHUR POLYNOMIALS

Like we used Schubert polynomials to write down Schubert varieties in the flag manifold we now define Schur polynomials to establish an analog in the Grassmannian. In particular, we are interested what the inclusion $H^*(Gr_{k,n}) \to H^*(F\,l_n)$ looks like on the level of polynomials.

Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ of lenght k. We will always assume $\lambda_1 \leq n-k$. With this partition we associate a Young diagram with at most k rows and by abuse of notation we also call it λ (compare with Definition 3). We define a symmetric polynomial $s_\lambda(x_1,\ldots,x_k)$ as follows. For a *numbering* T of the Young diagram λ we define

$$x^T = \prod_{i=1}^{k}(x_i)^{\text{number of times } i \text{ occurs in } T}.$$

The *Schur polynomial* $s_\lambda(x_1,\ldots,x_k)$ is then given by

$$s_\lambda(x_1,\ldots,x_k) = \sum x^T$$

where the sum is over all possible numberings of the Young diagram λ with integers from 1 to k. It is a fact that these polynomials are symmetric and form a basis of the ring of symmetric polynomials $\mathbb{Z}[x_1,\ldots,x_k]^{S_k}$. Also, $I_{n,k} = \{s_\lambda : \lambda_1 \geq n-k\}$ is an ideal. Let $A_{n,k}$ be the quotient ring $\mathbb{Z}[x_1,\ldots,x_k]^{S_k}/I_{n,k}$.

Although the definitions of Schur polynomials and Schubert polynomials are quite different, it turns out that the former are special instances of the latter. Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ define a permutation $w \in S_n$ by

$$w = w(\lambda) = (\lambda_k + 1, \lambda_{k-1} + 2, \ldots, \lambda_1 + k, \text{[remaining elements in increasing order]}).$$

Since we assumed $\lambda_1 \leq n-k$ this is well defined. For example, if $\lambda = (3,2,1,1)$ then $w(\lambda) = (2,3,5,7,1,4,6)$. Note that in general the permutation $w(\lambda)$ has exactly one *descent* at k, i.e. $w(i) < w(i+1)$ if and only if $i = k$. On the other hand, every permutation w with exactly one descent at k defines a partition λ by $\lambda(k+1-i) = w(i) - i$ and these two transformations are inverses of each other. We will see later that $\mathfrak{S}_{w(\lambda)} = s_\lambda$ for every permutation λ of length k with $\lambda_1 \leq n-k$. So we call a Schubert variety X_w *Grassmannian* if $w = w(\lambda)$ for some permutation λ with $\lambda_1 \leq n-k$.

8. TRANSLATION BETWEEN GRASSMANNIAN AND FLAG MANIFOLD

To understand the injection $H^*(Gr_{k,n}) \to H^*(Fl_n)$ on the level of polynomials we need to establish a connection between these rings and the quotient ring $A_{n,k}$ of Schur polynomials and the quotient ring H_n of Schubert polynomials, respectively. We begin with the latter.

On Fl_n we have a filtration of the trivial bundle E_{Fl_n} given by

$$0 = U_0 \subset U_1 \subset \cdots \subset U_n = E_{Fl_n}.$$

Here, the fiber of U_i over the point $E_\bullet$ is simply given by E_i. By taking quotients we end up with n line bundles $L_i = U_i/U_{i-1}$ with base space Fl_n. Our desired isomorphism is given in terms of Chern classes. For the definition and details about Chern classes see Chapter 14 in [5].

Proposition 16. *The first Chern classes $c_1(L_i) \in H^2(Fl_n)$ of these line bundles generate the cohomology ring of Fl_n. More general, the map $H_n \to H^*(Fl_n)$ given by $x_i \mapsto c_1(L_i)$ is an isomorphism of rings. Furthermore, this map induces a one to one correspondence between Schubert polynomials and Schubert varieties, namely $\mathfrak{S}_w \leftrightarrow [X_w]$.*

Proof. Let e_i denote the ith symmetric polynomial in $\mathbb{C}[x_1,\ldots,x_n]$, so e_i is the sum over all monomials of degree i. By Proposition 10.3 in [3] the Chern classes $c_1(L_i)$ generate $H^*(Fl_n)$, subject to the relations $e_i(c_1(L_1),\ldots,c_1(L_n))$. Since the ith symmetric polynomials e_i generate the ideal S of symmetric polynomials in $\mathbb{C}[x_1,\ldots,x_n]$ the map is an isomorphism.

In [1] it is shown that $[X_w(F_\bullet)] = \partial_{w^{-1}w_0}[\{F_\bullet\}]$ where we define $\partial_u[X_v] = [X_{vu^{-1}}]$ for arbitrary permutations $u,v \in S_n$. By the definition of the Schubert polynomials it suffices to check $\pi^*(x_1^{n-1}x_2^{n-2}\cdots x_{n-1}) = [\{F_\bullet\}]$. This can be verified by an explicit calculation. $\square$

Quite similiar to the isomorphism between Schubert polynomials and Schubert varieties in the flag manifold we can write down the corresponding isomorphism for the Grassmannian. Let $d_1, \dots, d_k$ denote the Chern classes of the tautological k-plane bundle of the Grassmannian $Gr_{k,n}$. Then, by [6], the map $A_{n,k} \to H^*(Gr_{k,n}), x_i \mapsto d_i$ is an isomorphism of rings. We can state the even stronger result which can be found in [3].

Proposition 17. *The isomorphism between the quotient ring $A_{n,k}$ of Schur polynomials and the cohomology ring $H^*(Gr_{k,n})$ induces a one to one correspondence between Schur polynomials and Schubert varieties in the Grassmannian given by $s_\lambda \leftrightarrow [\Omega_\lambda]$.*

Now we are in a good position to answer the question what we mean by the statement

Flag manifolds are a generalization of Grassmannian manifolds.

Informally the answer is that every question in Schubert calculus stated in terms of Grassmannians can be translated into a question concerning flag manifolds. We start out on the level of polynomials.

Proposition 18. *Let λ be a permutation of length k with $\lambda_1 \leq n-k$. Then*

$$\mathfrak{S}_{w(\lambda)} = s_\lambda.$$

We prove this very nontrivial result using the following two facts for those proofs we refer to Lemma 10.12 in [3] and [4], respectively. The proof of the first is elementary. The proof of the second needs some amout of work. Originally, the expression in Lemma 20 was the definition of the Schur polynomials.

Lemma 19. *For $w_0 = (k\ \ k-1 \dots 2\ \ 1) \in S_k$, it holds that*

$$\partial_{w_0} = \frac{1}{\Delta} \sum_{w \in S_n} sgn(w) \cdot w,$$

where $\Delta = \prod_{i<j}(x_i - x_j)$ is the Vandermonde determinant.

Lemma 20. *(Jacobi-Trudi formula)*
For a partition λ, $s_\lambda(x_1, \dots, x_k) = \frac{det[x_j^{\lambda_i+k-i}]_{1\leq i,j\leq k}}{det[x_j^{k-i}]_{1\leq i,j\leq k}}$. Note, that the denominator is equal to the the Δ above.

Proof. (of Proposition 18) Let $u \in S_n$ with $u(i+1) < u(i)$ for $i < k$ and $u(i+1) > u(i)$ if $i \geq k$. Let $\delta_{a,b}\partial_i = \partial_i$ if $a > b$ and the identity operator otherwise. A decomposition of u with minimal length is

$$\begin{aligned} u =& u_0 \cdot [\delta_{n,u(1)} s_1] \cdot [\delta_{n-1,u(2)} s_2] \cdots [\delta_{n-k+1,u(k)} s_(k)] \cdot s_{k+1} \cdot s_{k+2} \cdots s_{n-1} \cdot \\ & [\delta_{n-1,u(1)} s_1] \cdot [\delta_{n-2,u(2)} s_2] \cdots [\delta_{n-k,u(k)} s_(k)] \cdot s_{k+1} \cdot s_{k+2} \cdots s_{n-2} \\ & \cdots \\ & [\delta_{1,u(1)} s_1]. \end{aligned}$$

A simple example should clear this weird representation up. Applying rule $a)$ in Lemma 14 leads to $\mathfrak{S}_u = x_1^{u(1)-1} x_2^{(u(2)-1)} \cdots x_k^{u(k)-1}$.

Let $u = (k\ \ k-1\ldots 2\ \ 1) \in S_n)$ and let $w' = w \cdot u$ where $w = w(\lambda)$. By the first part of the proof we then have

$$\begin{aligned}\mathfrak{S}_{w'} &= x_1^{w(k)-1} x_2^{(w(k-2)-1)} \cdots x_k^{w(1)-1}\\ &= x_1^{\lambda_1+r-1} x_2^{\lambda_2+r-2} \cdots x_k^{\lambda_k}.\end{aligned}$$

Again, by part $a)$ of Lemma 14, $\mathfrak{S}_w = \partial_u(\mathfrak{S}_{w'})$. By the Leibniz formula for determinants, $\sum_{w \in S_k} \operatorname{sgn}(w) \cdot w\mathfrak{S}_{w'} = \det[x_j^{\lambda_i+k-i}]_{1 \le i,j \le k}$. Using Lemma 19 and 20 we conclude

$$\mathfrak{S}_w = \frac{\det[x_j^{\lambda_i+k-i}]_{1 \le i,j \le k}}{\Delta} = s_\lambda(x_1, \ldots, x_k).$$

☐

Finally, we can summerize the relation between Schubert calculus in the Grassmannian and in the flag manifold. The former is just a special case of the latter. The translation is done by the canonical map π^* on cohomology.

Theorem 21. *Let* $\pi^* : H^*(Gr_{k,n}) \to H^*(Fl_n)$ *be the map on cohomology induced by the map* $\pi : Fl_n \to Gr_{k,n}$, $F_\bullet \mapsto F_k$. *Then* π^* *is given by* $\pi^*([\Omega_\lambda]) = [X_{w(\lambda)}]$. *Equivalently,* $\pi^*(s_\lambda) = \mathfrak{S}_{w(\lambda)}$. *On the level of polynomials,* π^* *is the identity map.*

Proof. The equality $\pi^*(s_\lambda) = \mathfrak{S}_{w(\lambda)}$ is precisely Proposition 10.9 in [3]. The equivalence follows from Propositions 16 and 17. The last statement follows from Proposition 18. ☐

In order to appy Proposition 10.9 in [3] we should note that our Schubert varieties X_w are called *dual* Schubert varieties in [3] and are denoted by Ω_w.

Caution 22. Although $\pi^* : H^*(Gr_{k,n}) \to H^*(Fl_n)$ acts as identity on polynomials, sending Schur polynomials, which are symmetric, into a ring which is a quotient by the ideal of symmetric polynomials, this map is still injective. Note that the Schur polynomial s_λ is a polynomial in k variables and symmetric in those. But considered as a polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$ it is very far from being symmetric and hence is not sent to 0 by π^*.

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