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Rank and Orbits of $2 \times 2 \times 2$ Matrices

Projektarbeit
von
Florian Block und Patrick Schmid

Themensteller:	Prof. Dr. Jürgen Richter-Gebert
Betreuer:	Prof. Dr. Jürgen Richter-Gebert
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1 Motivation

Quantum Entanglement has been in the center of a vast area of research in recent years. The basic task is to find the orbits of certain systems of quantum bits under certain operations. Quantum bits (or *qubits*) are objects described by a superposition of two *base states*, say $|0\rangle$ and $|1\rangle$, having complex amplitudes a_0 and a_1:

$$|\varphi\rangle = a_0 \cdot |0\rangle + a_1 \cdot |1\rangle.$$

A state of a single qubit can be interpreted as a unit vector in the two dimensional complex Hilbert space $\mathcal{H} = \mathbb{C}^2$. We require $|a_0|^2 + |a_1|^2 = 1$, since the squares of the absolute values of the amplitudes have an interpretation as observation probabilities of the base states. The states $|0\rangle$ and $|1\rangle$ form an orthonormal basis and can be identified with the two standard basis vectors of $\mathcal{H}$. The brackets $|\cdot\rangle$ come from the standard notation in quantum mechanics and can be ignored safely if one keeps in mind that $|0\rangle$ and $|1\rangle$ are formal base states. Concrete examples of physical objects having all properties of qubits are electrons or photons.

The state space of a system of qubits is the tensor product of their state spaces. For a 3 qubit[1] system this space is spanned by

$$|000\rangle := |0\rangle \otimes |0\rangle \otimes |0\rangle,\ |001\rangle := |0\rangle \otimes |0\rangle \otimes |1\rangle,\ \ldots\ , |111\rangle := |1\rangle \otimes |1\rangle \otimes |1\rangle.$$

Therefore, a general state can be written as

$$|\varphi\rangle = a_{000} \cdot |000\rangle + a_{001} \cdot |001\rangle + a_{010} \cdot |010\rangle + \cdots + a_{111} \cdot |111\rangle.$$

We again require $|a_{000}|^2 + |a_{001}|^2 + \cdots + |a_{111}|^2 = 1$. The 3 qubit system is totally determined by the 8 complex numbers $a_{000}, a_{001}, \ldots, a_{111}$.

There are quite a lot of very comprehensible introductions into the theory of quantum computation and quantum entanglement. See, for example, [1] [4] [5].

A *global* operation on a system of qubits is any change of the amplitudes a_{ijk} still satisfying the probability condition. The mathematical description of a *local* operation on the system is motivated by the physical action of changing the state of *one* qubit, say an electron in a system of 3 electrons. The surprising fact: Given a certain system of 3 qubits, that is the amplitudes a_{ijk}, it is *impossible* to transform the system only by local operations into all states it could potentially be transformed by global operations. This gap does not exist in the world of classic bits. Of course, by changing a letter at a time in a sequence of classic bits with fixed length one is able to end up with every

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[1] For notational convenience we consider only systems of 3 qubits since we restrict ourselves to this case later on anyway. The general n qubit case ($n > 1$) works analogously.

possible 0-1-word of that length. In the quantum world there is "something" that is invariant under local transformations. This "something" is quantum entanglement.

To make this rather vague notion of quantum entanglement more precise, we call our system of 3 qubits *unentangled* if

$$\begin{aligned}|\varphi\rangle &= (a_0 \cdot |0\rangle + a_1 \cdot |1\rangle) \otimes (b_0 \cdot |0\rangle + b_1 \cdot |1\rangle) \otimes (c_0 \cdot |0\rangle + c_1 \cdot |1\rangle) \\ &= |\varphi_a\rangle \otimes |\varphi_b\rangle \otimes |\varphi_c\rangle.\end{aligned}$$

In some sense this means that the state of each qubit is "separated" from the states of other qubits, each qubit has a state of its "own". Quantum entanglement seems to be the source of the power of quantum computation which, in principle, is more powerful than any classic Turing machine. Further details about quantum computation and quantum information and quantum algorithms can be found in [5].

A local operation is a triple $M = (A, B, C)$ of invertible complex 2×2 matrices. It acts on an unentangled system in the obvious way:

$$M(|\varphi\rangle) = A(|\varphi_a\rangle) \otimes B(|\varphi_b\rangle) \otimes C(|\varphi_c\rangle)$$

where $A(|\varphi_a\rangle)$ is standard matrix vector multiplication in the Hilbert space $\mathcal{H}$.

Every quantum state is a sum of unentangled states. By linearity we define a local operation on a general quantum state by

$$M(|\varphi\rangle) = \sum M(|\varphi_i\rangle)$$

for $|\varphi\rangle = \sum |\varphi_i\rangle$. Distributivity assures this definition to be well defined.

The fundamental task is to *determine the orbits of this operation* since they measure some level of entanglement of the system. A different (and more difficult) problem is to answer the same question for the operation $SL_2(\mathbb{C}) \times SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$ defined in the same way instead of $GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$. Although the former is physically more relevant, the orbits of the latter are still a coarsening of the orbits of the former. So the knowledge of the orbits of the $GL_2(\mathbb{C})^3$-action still reveals some information about different levels of quantum entanglement.

We will answer the basic question of the last paragraph for 3 qubit systems and the $GL_2(\mathbb{C})^3$-action. Our approach is to represent a general state of the system by a complex $2 \times 2 \times 2$ (hyper)matrix with entries a_{ijk} in the obvious fashion. A local operation, say, on the first qubit is given by (A, I_2, I_2) where $A \in GL_2(\mathbb{C})$ and I_2 is the identity matrix. It turns out that this operation is equivalent to applying the matrix A to each of the four columns of length two that is aligned with the direction corresponding to the first qubit.

Not only are orbits (by definition) invariants of the $GL_2(\mathbb{C})^3$-action, but also the *rank* of a hypermatrix which we define and investigate later on. It is a generalization of the usual notion of rank for ordinary matrices. It turns out that this measure of entanglement is even coarser than the orbits of the $GL_2(\mathbb{C})^3$-action. The so called *rank-1-matrices* that will appear later on correspond exactly to unentangled quantum systems.

2 Introduction

2.1 Fundamentals

To make sure that we all talk about the same structure and to introduce some notation, which makes our discussion easier, we introduce hypermatrices by the following

Definition 2.1 *1. Let $k \in \mathbb{N}$, $n_1, \ldots, n_k \in \mathbb{N}$ and F be a field. Then we call*

$$A = (a_{i_1 \cdots i_k})_{1 \leq i_j \leq n_j; 1 \leq j \leq k}$$

a hypermatrix of size $n_1 \times \cdots \times n_k$ over the field F. *We call k the* spatial dimension *of A. We will write $F^{n_1 \times \cdots \times n_k}$ for all hypermatrices of size $n_1 \times \cdots \times n_k$ over the field F. If $f = (n_1, \ldots, n_k)$ (with k a positive integer) is a format (k-tuple), we write F^f instead of $F^{n_1 \times \cdots \times n_k}$.*

2. The sum of two hypermatices of the same size is defined componentwise.

In our development of the rank of hypermatrices we will always consider $F = \mathbb{C}$. But everything in Section 2 to 4 is true for every field. The following definition and different facts about (hyper-)rank can be found in [2].

Definition 2.2 *A hypermatrix A of size $n_1 \times \cdots \times n_k$ is a* rank-1-matrix *if there exist $p_j \in \mathbb{C}^{n_j} \backslash \{0\}$ for $1 \leq j \leq k$,$1 \leq i_j \leq n_j$ such that*

$$a_{i_1 \cdots i_k} = p_{1,i_1} \cdot \cdots \cdot p_{k,i_k}$$

where $p_{j,i_j} = (p_j)_{i_j}$ for $1 \leq j \leq k$ and $A = (a_{i_1 \cdots i_k})_{1 \leq i_j \leq n_j; 1 \leq j \leq k}$.

Now we are in a good position to define the (hyper-)rank of a general hypermatrix.

Definition 2.3 *Let A be a hypermatrix. Then the* (hyper-)rank *r of A is the smallest integer such that A is the sum of r rank-1-matrices of the same size as A. The (hyper-)rank r of the zero-matrix is 0 by definition. We write $r = rank(A)$.*

In fact, this rank is a generalization of the ordinary rank of a $m \times n$-matrix A, that is the dimension of the image of the corresponding canonical linear map $\Phi_A, \mathbb{C}^n \to \mathbb{C}^m, x \mapsto Ax$. To distinguish for the moment between hyperrank and ordinary rank we write $ordrank(A) := dim(Im(\Phi_A))$.

Lemma 2.4 *Let A be an $m \times n$-matrix. Then the hyperrank of A is the same as the ordinary rank of A.*

Proof. In Lemma 3.5 we proof that $hyperrank(A)$ is invariant under Gaussian elimination steps. Because this is also true for $ordrank(A)$ we can assume that A is in upper triangular form. Let $a_1, \ldots, a_m$ denote the rows of A. For short we write $r := ordrank(A)$ and $s := hyperrank(A)$. We have $A = a_1 e_1^T + \cdots + a_r e_r^T$ where e_i denotes the $i-th$ standard basis vector of $\mathbb{C}^n$. Because $a_i e_i^T$ is a rank-1-matrix for every i this implies $hyperrank(A) \le ordrank(A)$.
Let $A = A_1 + \cdots + A_s$ be a minimal rank-1-representation. Then it holds $ordrank(A) = ordrank(A_1 + \cdots + A_s) \le ordrank(A_1) + \cdots + ordrank(A_s) = s = hyperrank(A)$ and the lemma is proved. □

We have shown that there is no difference between hyperrank and ordinary rank of $m \times n$-matrices. This fact allows us to use the term *rank* instead of hyperrank.

The propositions which follow are simple consequences of the definitions given above:

Proposition 2.5 *Let $A \in \mathbb{C}^{\overbrace{2 \times \cdots \times 2}^{n}}$. Then it holds:*

$$rank(A) \le 2^{n-1}.$$

Proof. "Cutting and adding the slices." □

Proposition 2.6 *Let A and B be two hypermatices of the same size. Then the following inequation holds:*

$$rank(A + B) \le rank(A) + rank(B).$$

Proof. Obvious. □

Now we will define an operation which is very useful for building up hypermatrices from "smaller objects" and for determining the rank of an arbitrary $2 \times \cdots \times 2$ matrix.

Definition 2.7 *Let $k \in \mathbb{N}$, $n_1, \ldots, n_k \in \mathbb{N}$ and $l \in \mathbb{N}$. We define:*

$$\dot{\otimes} : \mathbb{C}^{n_1 \times \cdots \times n_k} \times \mathbb{C}^l \to \mathbb{C}^{n_1 \times \cdots \times n_k \times l}, \quad (A, v) \mapsto A \dot{\otimes} v$$

where

$$(A \dot{\otimes} v)_{i_1 \cdots i_k i} := a_{i_1} \cdots a_{i_k} v_i$$

for $1 \leq i_j \leq n_j$, $1 \leq j \leq k$ *and* $1 \leq i \leq l$ *and with the usual notation* $A = (a_{i_1 \cdots i_k}), v = (v_1, \ldots, v_l)^T$. *This map will be called the (matrix-vector) tensor-product.*

Properties of the tensor-product $\dot{\otimes}$

Proposition 2.8 *1. Let* A *be a* $n_1 \times \cdots \times n_k$ *hypermatrix. Then* A *is a rank-1-matrix if and only if it can be written as*

$$A = v_1 \dot{\otimes} \cdots \dot{\otimes} v_k$$

where $v_j \in \mathbb{C}^{n_j} \backslash \{0\}$ *for* $1 \leq j \leq k$.

2. If A *is a rank-1-matrix and* $w \in \mathbb{C}^l \backslash \{0\}$ *for some* $l \in \mathbb{N}$, *then* $A \dot{\otimes} v$ *is also a rank-1-matrix.*

Proof. Again obvious. □

Proposition 2.9 *Let* $A \in \mathbb{C}^f$ *be a complex hypermatrix and* $v \in \mathbb{C}^l \backslash \{0\}$ *a complex vector. Then:*

$$rank(A) = rank(A \dot{\otimes} v).$$

Proof. If $A = 0$ then the result follows easily. Otherwise let $r := rank(A)$ and $A_1, \ldots, A_r$ be rank-1-hypermatrices of the same size as A such that $A = A_1 + \cdots + A_r$. It follows: $A \dot{\otimes} v = (A_1 + \cdots + A_r) \dot{\otimes} v = A_1 \dot{\otimes} v + \cdots + A_r \dot{\otimes} v$. $A_i \dot{\otimes} v$ is a rank-1-matrix for all i by Proposition 2.8. Therefore $rank(A \dot{\otimes} v) \leq r$.
Let $s := rank(A \dot{\otimes} v)$ and $i_0 \in \{1, \ldots, l\}$ such that $v_{i_0} \neq 0$. Consider $A \dot{\otimes} v = \hat{A}_1 + \cdots + \hat{A}_s$, a minimal rank 1 representation of $A \dot{\otimes} v$. Because $\hat{A}_i$ is a rank-1-matrix for every i, Proposition 2.8 tells us that there exist a rank-1-matrix $A'_i \in \mathbb{C}^f$ and a vector $w'_i \in \mathbb{C}^l$ for every i such that $\hat{A}_i = A'_i \dot{\otimes} w'_i$ for every i. Let η_i denote the i_0-th component of w'_i. Then we have $A = (v_{i_0})^{-1}(A'_1 \eta_1 + \cdots + A'_s \eta_s)$. Because the A'_i's are rank-1-matrices, we have $rank(A) \leq s = rank(A \dot{\otimes} v)$. □

2.2 Permutation Invariance of Rank

To complete this paragraph we show that the rank of a hypermatrix is invariant under permutation of coordinates. For simplicity, we consider only the case of $2 \times \cdots \times 2$ matrices. The general case works analogously.

Definition 2.10 *Let* $\pi \in S_k$ *a permutation of* $\{1, \ldots, k\}$. *We define*

$$\tilde{\pi} : \Lambda^n := \mathbb{C}^{\overbrace{2 \times \cdots \times 2}^{n}} \to \Lambda^n,$$

$$A = (a_{i_1, \ldots, i_n})_{1 \le i_1, \ldots, i_n \le 2} \mapsto \tilde{\pi}(A)$$

where

$$\tilde{\pi}(A)_{1_1 \cdots i_n} := a_{\pi(i_1)} \cdots a_{\pi(i_n)} \qquad 1 \le i_1, \ldots, i_n \le 2.$$

We also write $\pi(A)$ *for* $\tilde{\pi}(A)$.

Proposition 2.11 *For every* $\pi \in S_n$ *the function* $\tilde{\pi}$ *is linear.*

Proof. Let $A, B \in \Lambda^n$ and $1 \le i_1, \ldots, i_n \le 2$. Then we have

$$\pi(A+B)_{i_1, \ldots, i_n} = (A+B)_{\pi(i_1), \ldots, \pi(i_n)}$$

$$= (A)_{\pi(i_1), \ldots, \pi(i_n)} + (B)_{\pi(i_1), \ldots, \pi(i_n)} = \pi(A)_{i_1, \ldots, i_n} + \pi(B)_{i_1, \ldots, i_n}.$$

This implies $\pi(A+B) = \pi(A) + \pi(B)$. □

Lemma 2.12 *Let* $A \in \Lambda^n$ *and* $\pi \in S_n$. *The following equation holds:*

$$rank(\pi(A)) = rank(A).$$

Proof. For rank-1-matrices the proposed equation is true by definition. If $A = 0$ then the result follows. Otherwise let $r := rank(A) > 0$ and $A_1, \ldots, A_r \in \Lambda^n$ rank-1-matrices such that $A = A_1 + \cdots + A_r$ $(*)$. Suppose $rank(\pi(A)) > rank(A) = r$. Then we have

$$\pi(A) = \pi(A_1 + \cdots + A_r) = \pi(A_1) + \cdots + \pi(A_r)$$

which is a contratiction to $(*)$ since the $\pi(A_i)$ are rank-1-matrices. Therefore we have $rank(\pi(A)) \le rank(A)$. Since S_n is a finite group there exists $N \in \mathbb{N}$ such that $\pi^N = id_{S_n}$. We get

$$rank(A) = rank(\pi^N(A)) \le \mathrm{rank}(\pi^{(N-1)}(A)) \le \cdots \le rank(\pi(A))$$

and the lemma follows. □

3 Local Operations and Rank Invariance

3.1 Local Operations

Now we define the general local operation on a hypermatrix. We start with the definition for local operations on a rank-1-matrix.

Definition 3.1 *Let $f = (n_1, \dots, n_k)$ be a format. We call the set*

$$L_f(\mathbb{C}) := \bigotimes_{i=1}^{k} GL_{n_i}(\mathbb{C}) := \{M_1 \otimes \cdots \otimes M_k; M_i \in GL_{n_i} \text{ for } i = 1, \dots, k\}$$

together with the operation

$$\odot : L_f(\mathbb{C}) \times L_f(\mathbb{C}) \to L_f(\mathbb{C}),$$

$$(M_1 \otimes \cdots \otimes M_k, N_1 \otimes \cdots \otimes N_k) \mapsto M_1N_1 \otimes \cdots \otimes M_kN_k$$

the group of local operations of format f *(here $\bigotimes$ is just a formal notation).*

It is clear that $(L_f(\mathbb{C}), \odot)$ is a group. Furthermore, this group induces a canonical operation on the set of hypermatrices of format f. We start defining the operation of $L_f(\mathbb{C})$ on $\mathbb{C}^f$ by defining it at first for rank-1-matrices.

Definition 3.2 *Let $A \in \mathbb{C}^f$ be a rank-1-matrix, say $A = v_1 \otimes \cdots \otimes v_k$, where $v_i \in \mathbb{C}^{n_i}$ for all i. Let $M \in L_f(\mathbb{C})$, say $M = M_1 \otimes \cdots \otimes M_k$, where $M_i \in GL_{n_i}(\mathbb{C})$ for all i. Then we define the* (local) operation of M on A *as*

$$M(A) := \bigotimes_{i=1}^{k} M_i v_i := M_1v_1 \otimes \cdots \otimes M_kv_k.$$

This leads to the following natural extension of our operation.

Definition 3.3 *Let $A \in \mathbb{C}^f$ and $A = A_1 + \cdots + A_s$ a rank-1-representation of A. Then we define the operation of $M \in L_f(\mathbb{C})$ on A by*

$$M(A) := \sum_{i=1}^{s} M(A_i).$$

Remark 3.4 *The definition above is independent of the specific rank-1-representation because of distributivity of the ordinary matrix-vector multiplication.*

3.2 Rank Invariance

Lemma 3.5 *Let $A \in \mathbb{C}^f$ and $M \in L_f(\mathbb{C})$. Then :*

$$rank(M(A)) = rank(A).$$

Proof. Let A be a rank-1-matrix, namely $A = v_1 \otimes \cdots \otimes v_k$ with $v_i \in (\mathbb{C}^{n_i})^*$, and $M = M_1 \otimes \cdots \otimes M_k$ with $M_i \in GL_{n_i}(\mathbb{C})$. Since M_i is invertible for all i and $v_i \neq 0$ for all i we have $M_i v_i \neq 0$ for all i. This implies that $M(A) = M_1 v_1 \otimes \cdots \otimes M_k v_k$ is a rank-1-matrix. That means that our statement is true for rank-1-matrices.
Let A be an arbitrary matrix of format f and $A = A_1 + \cdots + A_r$ a minimal rank-1-representation of A. Then we have by definition $M(A) = M(A_1) + \cdots + M(A_r)$. Because the statement is true for rank-1-matrices we obtain $rank(M(A)) \leq r = rank(A)$. This means that the rank of a matrix cannot increase by applying a local operation. Every local operation is invertible and therefore we have $rank(A) = rank(M^{-1}(M(A))) \leq rank(M(A))$. $\square$

The geometrical interpretation.
Let M be a local operation of the form $M = I \otimes \cdots \otimes I \otimes M_{i_0} \otimes I \otimes \cdots \otimes I$ where I is the identity matrix and let A be a fitting hypermatrix. We want to find a geometric interpretation for the operation of M on A.
We first analyse what happens if A is a rank-1-matrix. Let $v_i \in (\mathbb{C}^{n_i})^*$ such that $A = v_1 \otimes \cdots \otimes v_k$. By definition we have

$$M(A) = v_1 \otimes \cdots \otimes v_{i_0-1} \otimes M_{i_0} v_{i_0} \otimes v_{i_0+1} \otimes \cdots \otimes v_k.$$

That means that the operation M on A is equivalent to the following action on A: Take each $1 \times \cdots \times 1 \times n_{i_0} \times 1 \times \cdots \times 1$ submatrix which points in the direction i_0, inteprete this submatrices as vectors in $\mathbb{C}^{n_{i_0}}$, multiply each of them with M_{i_0} and put them together again in the same manner. See Figure 1 for an illustration.

If A is an arbitrary matrix and $A = A_1 + \cdots + A_s$ is a rank-1-representation of A then the above geometrical interpretation is obviously true for every A_i if we apply M to A_i. But because matrix-vector-multiplication is distributive the same geometrical interpretation holds for an arbitrary matrix A.

3.3 Generating Systems

We introduce the following notation: We write $E_{ij}^{(n)}(\lambda) \in GL_n(\mathbb{C})$ ($1 \leq i, j \leq n, \lambda \in \mathbb{C}$) for the elementary (ordinary) matrix, which, multiplied from the left to an (ordinary) matrix with fitting size, corresponds to the Gaussian

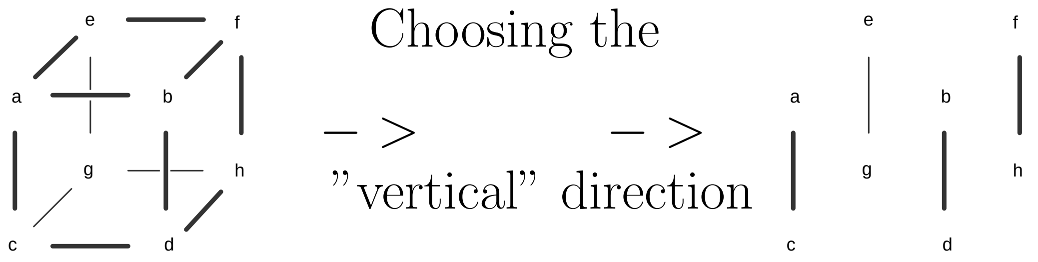


Figure 1: Identifying the submatrices: here in a case with a three dimensional hypermatrix the submatrices are vectors with two elements which perform the well known matrix-vector multiplication.

operation "Add λ times the i^{th} row to the j^{th}". Further let $S_i^{(n)}(\mu) \in GL_n(\mathbb{C})$ $(1 \leq i \leq n, \mu \in \mathbb{C}\backslash\{0\})$ be the scaling matrix, which, applied in the same way just described, corresponds to the Gaussion operation "Multiply the i^{th} row with $\mu \neq 0$". Finally, let $P_{ij}^{(n)} \in GL_n(\mathbb{C})$ $(1 \leq i, j \leq n)$ be a permuation matrix, which, again applied in the same way, exchanges the i^{th} and j^{th} row of a given matrix and therefore corresponds to the third Gaussian operation.

Remark 3.6 *Since on the one hand every matrix in $GL_n(\mathbb{C})$ can be transformed by Gaussian operations into the identity matrix I_n and clearly $I_n = P_{ii}^{(n)}$ and on the other hand every Gaussian operation is invertible, we have the following generating property:*

$$< \{E_{ij}^{(n)}(\lambda)\} \cup \{S_i^{(n)}(\mu)\} \cup \{P_{ij}^{(n)}\} >= GL_n(\mathbb{C}).$$

3.4 Gaussian Elimination on Hypermatrices

Keeping Remark 3.6 in mind, it is very helpful to generalize the well-known Gaussian steps for 2-dimensional matrices to hypermatrices of format f when looking for orbits under $L_f(\mathbb{C})$.

Definition 3.7 *We call every element $M \in L_f(\mathbb{C})$ an* elementary (Gaussian) operation *(on a matrix of format f) or* Gaussian step *if M is of one of the following forms*

a) $M = I \otimes \cdots \otimes I \otimes E_{ij}^{(n)}(\lambda) \otimes I \otimes \cdots \otimes I$

b) $M = I \otimes \cdots \otimes I \otimes S_i^{(n)}(\mu) \otimes I \otimes \cdots \otimes I$

c) $M = I \otimes \cdots \otimes I \otimes P_{ij}^{(n)} \otimes I \otimes \cdots \otimes I$

with a fitting n.

The elementary operations on hypermatrices are very useful for practical calculations because they are quite easy to compute and by Remark 3.6 they provide all information about orbits under $L_f(\mathbb{C})$.

All Gaussian steps have a very descriptive geometrical interpretation close to the well known Gaussian operations on $m \times n$-matrices. If M is a elementary Gaussian operation and if l is the "position" of the non-identity matrix in the notation of Definition 3.7, then M operates on a hypermatrix $A \in \mathbb{C}^f$ in the following manner:
If M is of form a) then M corresponds to the action "Add λ times the i^{th} slice of A to the j^{th}", where we enumerate the slices in the l^{th} direction.
If M is of form b) then M corresponds to the action "Multiply the i^{th} slice of A with $\mu \neq 0$", where we enumerate the slices in the l^{th} direction.
If M is of form c) then M corresponds to the action "Exchange the i^{th} and the j^{th} slice of A", where we enumerate the slices in the l^{th} direction.

4 A Different View on Multiplication

In this section we are going to develop a "general multiplication" of which many known multiplications are only special cases. Especially local operations on a hypermatrix can be embedded in this general theory.

4.1 A Collection of Known Multiplications

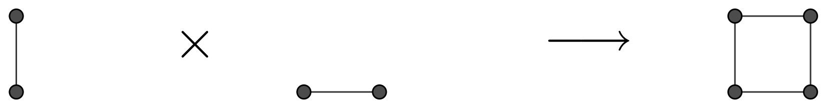
We start by listing some known multiplications and illustrate them by pictures which will lead to a better understanding of the definition of the "general multiplication". Again, we abbreviate $\Lambda^n = \mathbb{C}^{\underbrace{2\times\cdots\times 2}_{n}}$.

1. $\mathbb{C}^1 \times \mathbb{C}^1 \to \mathbb{C}^1$,
   $(z, w) \mapsto zw$.

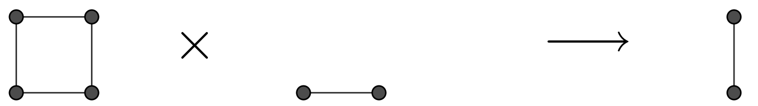
2. $\mathbb{C}^1 \times \mathbb{C}^2 \to \mathbb{C}^2$,
   $(\lambda, v) \mapsto \lambda v$ where $(\lambda v)_i = \lambda v_i$.

3. $\mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$,
   $(v, w) \mapsto v^T w = \sum_{i=1}^{2} v_i w_i$.

4. $\mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^{2\times 2}$,
   $(v, w) \mapsto vw^T$.



5. $\mathbb{C}^{2\times 2} \times \mathbb{C}^2 \to \mathbb{C}^2$,
   $(A, v) \mapsto Av$ where $(Av)_i = \sum_{j=1}^{2} a_{ij} v_j$.



6. $\mathbb{C}^{n\times n} \times \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$,
   $(A,B) \mapsto AB$ where $(AB)_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk}$.

7. $\Lambda^k \times \Lambda^1 \to \Lambda^{k+1}$,
   $(A,v) \mapsto A\dot{\otimes}v$ where $(A\dot{\otimes}v)_{i_1\cdots i_n i} = a_{i_1}\cdots a_{i_n}v_i$.

If we have a closer look at the pictures we realize two crucial facts:

1. Each "direction" which appears only in one of the input objects remains and becomes a "direction" in the final object.
2. If a "direction" appears in both input objects then it disappears. This "direction annihilation" turns out to be a sum over this "direction".

4.2 The General Multiplication

The geometric observation made above motivates the following

Definition 4.1

1. *Let $A \in \Lambda^n$ in the usual notation $A = (a_{i_1,\ldots,i_n})$. With A we associate an injective function $d_A : \{1,\ldots,n\} \to \mathbb{N}$ enumerating the spatial dimensions of A.*
2. *Let $n,m,k \in \mathbb{N}$, $k \le n,m$ and $A = (a_{i_1,\ldots,i_n}) \in \Lambda^n$, $B = (b_{i_1,\ldots,i_m}) \in \Lambda^m$. Furthermore for $1 \le j \le k$ let $d_A(n-k+j) = d_B(j)$ and $d_A(j_1) \ne d_B(j_2)$ for all $1 \le j_1 \le n-k$ and $m-k \le j_2 \le m$. (This means that the last k directions of A are the same as the first k directions of B and all other directions are different.)*
   Then we define

$$(A\tilde{\otimes}B)_{i_1,\ldots,i_r} := \sum_{j_1=1}^{2}\cdots\sum_{j_k=1}^{2} a_{i_1,\ldots,i_p,j_1,\ldots,j_k} b_{j_1,\ldots,j_k,i_{p+1},\ldots,i_r}$$

   where $r := n+m-2k$ is the dimension of $A\tilde{\otimes}B$ and $p := n-k$ is the number of directions of $A\tilde{\otimes}B$ which come from A. We call this operation the general multiplication *of A and B.*

Example. We get the known multiplications listed above by setting the parameters n, m, k as follows:

1. $n, m, k = 0$.
2. $n, k = 0, m = 1$.
3. $n, m, k = 1$.
4. $n, m = 1, k = 0$.
5. $n = 2, m, k = 1$.
6. $n, m = 2, k = 1$.
7. $n = l, m = 1, k = 0$.

■

Remark 4.2

1. *The last example showed us that there is in fact no difference between $\tilde{\otimes}$ and $\dot{\otimes}$ when acting on $\Lambda^n \times \mathbb{C}$ if k is set to zero. That is why we can identify these operations.*
2. *The cross product is not a special case of the general multiplication. That is because the dimension of $A \tilde{\otimes} B$ is always even if the dimensions of A and B have the same parity.*
3. *The generalization for arbitrary formats is straight forward.*

5 The $2 \times 2 \times 2$ Case

5.1 The Rank of $2 \times 2 \times 2$ Hypermatrices

Remark 5.1 *In the following discussion we will examine hypermatrices and hyperdeterminants, but often we will abbreviate these expressions and say just matrix or determinant. If we say GKZ-hyperdeterminant, we mean the hyperdeterminant according to the book of Gelfand, Kapranov and Zelevinski [3].*

Definition 5.2 *Given a hypermatrix* $A = (a_{a_1,\dots,a_n})$, $a_1 \in \Delta_1, \dots, a_n \in \Delta_n$, *$n$ a positive integer, and $\Delta_i, 1 \leq i \leq n$, a nonempty finite set, we call n the* spatial dimension. *Whereas Δ_i is called the* realization *of a particular spatial dimension.*

Remark 5.3 *To simplify the notation, we label the elements of a hypermatrix A with $n = 3$ and $\Delta_i = \{0, 1\}$ for all $1 \leq i \leq n$ like in Figure 2 . A hypermatrix of this kind will also be called $2 \times 2 \times 2$ hypermatrix.*

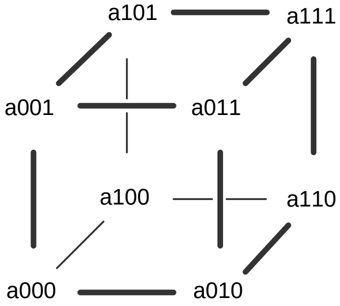


Figure 2: Labeling the elements

We will begin with a modification of these hypermatrices from which we will later construct a mutually exclusive and collectively exhaustive structure (in other words a tree).

Theorem 5.4 *A $2 \times 2 \times 2$ hypermatrix $A = (a_{jkl})$, $j, k, l \in \{0, 1\}$ with complex entries can be transformed by Gaussian steps in a form in which the front slice has two zero-elements on diagonal and the back slice has at least one zero entry which is located behind the other diagonal of the front slice, for example $a_{000} = a_{011} = a_{110} = 0$ (see the following figure) . This is called the* standard reduction (SR) *of a $2 \times 2 \times 2$ hypermatrix. The setting of the zero-elements is unique up to permutation of the slices.*

Proof. Since we have three possible spatial dimensions at least three zero-entries can be created in the following manner: Create two zeros in the front slice $\{a_{0kl}; k, l \in 0, 1\}$ on a diagonal (in the same way we would do it for a two dimensional matrix, so that a_{000} and a_{011} vanish) and one in the back slice $\{a_{1kl}; k, l \in 0, 1\}$ by addition of the front slice to the other so that an entry behind the diagonal of possible non-zero-entries (preferably a_{110}) vanishes. □

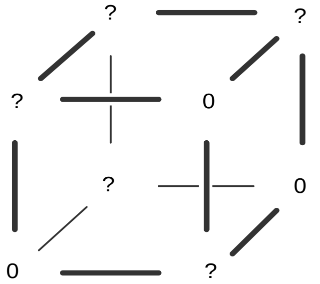


Figure 3: A standard reduction

Please note that the question marks in Figure 3 indicate possible non-zero-elements. The SR was for arbitrary hypermatrices. For the special structure of a rank-1-matrix we can state a stronger proposition.

Lemma 5.5 *A* $2 \times 2 \times 2$ *rank-1-hypermatrix can be transformed by Gaussian steps in a hypermatrix with just one non-zero-element (NZE).*

Proof. In the hypermatrix we have three spatial dimensions and for each spatial dimension two possible directions for the addition of slices. Without loss of generality the rank-1-hypermatrix has a slice in which all entries equal zero (otherwise add one slice with the appropriate scalar factor to the other). The reduction of a rank-1-matrix $B = (b_{jk})$, $j, k \in \{0, 1\}$ to a matrix with one NZE is trivial. □

Now we can state an analogous result for rank-2-matrices.

Theorem 5.6 *A* $2 \times 2 \times 2$ *rank-2-hypermatrix with complex entries can be transformed by Gaussian steps in a hypermatrix with only two non-zero-elements (NZEs).*

Proof. Assume that the $2 \times 2 \times 2$ rank-2-matrix has more than two non-zero elements. By definition a rank-2-matrix can be written as the sum of two rank

one matrices. By Lemma 5.5 a rank-1-matrix can be transformed by three Gaussian steps in a matrix which has only one non-zero-element. We have to decide whether the two rank-1-matrices can be transformed simultaneously in two matrices with one non-zero element.
This can be attained since we have six possibilities of adding one slice to another: Use arbitrarily three (each out of a different spatial dimension) for the first matrix and the other three for the second one. Note that the simultaneous Gaussian steps leave the rank of the second matrix invariant and that the last three Gaussian steps do not create a non-zero element in the first matrix since the possibilities of addition are chosen in a manner so that in the first already transformed matrix only zero slices are added. The addition of the two transformed matrices leads to the expected result. □

Keeping the shown results in mind, we define a special kind of matrix in which the only non-zero-elements are set like the coordinate axis in three dimensional space. The reduction of this special matrix to only two non-zero-elements is not obvious (of course, if such a reduction exists!).

Definition 5.7 *The matrix* $A = (a_{000}, a_{010}, a_{001}, a_{011}, a_{100}, a_{110}, a_{101}, a_{111}) = (1, 0, 0, 1, 0, 0, 1, 0)$ *is called the* independent matrix*. See Figure 4 for an illustration.* $2 \times 2 \times 2$ *hypermatrices which can be transformed into the independent by switching the slices are called* independent*.*

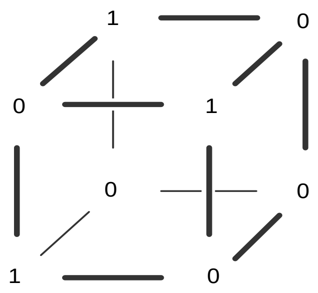


Figure 4: The independent matrix.

Wondering about the reduction to two non-zero-elements, we discover that such a reduction does not exist.

Proposition 5.8 *The independent matrix has rank 3.*

Proof. Assume that the independent matrix (call it N) can be written as the sum of two rank-1-matrices, let us call them A and B. These matrices have

both only NZEs, otherwise it would not be possible to generate the independent matrix. This can easily be seen if you look at the geometric interpretation (the cubes) of rank-1-matrices with some zero-elements.
Now look at an NZE in the independent matrix N and at the same position in the two rank-1-matrices A and B, for example n_{000}, a_{000} and b_{000}. For the ease of computation we scale both sides of the equation so that $a_{000} = 1$. Since A and B have rank 1 and only NZEs, there are unique scalars $\alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3$ with

$$a_{010} = \alpha_1, a_{100} = \alpha_2, a_{001} = \alpha_3$$

and

$$b_{010} = \beta_1 * b_{000}, b_{100} = \beta_1 * b_{000}, b_{001} = \beta_1 * b_{000}$$

Since $n_{010} = n_{100} = n_{001} = 0$ the following equations hold:

$$-\alpha_1 = \beta_1 * b_{000}, -\alpha_2 = \beta_2 * b_{000}, -\alpha_3 = \beta_3 * b_{000}$$

Now look at n_{110}; $n_{110} = 0$ so $a_{110} = -b_{110}$. Using again that A and B have rank 1, we get the equation

$$\alpha_1 * \alpha_2 = -\beta_1 * \beta_2 * b_{000}$$

If we plug in $-\alpha_1 = \beta_1 * b_{000}$ for example, we obtain $\alpha_2 = \beta_2$ and therefore $b_{000} = -1$ which is a contradiction. □

With these lemmas, we are in good position to find the answer for the question which ranks are actually attained by a hypermatrix. By now we are only aware of an upper bound.

Theorem 5.9 *A $2 \times 2 \times 2$ hypermatrix $A = (a_{jkl})$, $j, k, l \in \{0, 1\}$ with complex entries attains as values of rank zero, one, two and three.*

Proof. By definition a rank-0-matrix exists. The matrix A with $a_{jkl} = 1$ for all $j, k, l \in \{0, 1\}$ is a rank-1-matrix.
A rank-2-matrix is given as a matrix with only two non-zero-elements so that the first Gaussian step cannot annihilate a non-zero element, for example $a_{000} = a_{011} = 1$. This matrix has rank two since an upper bound is provided by the canonical decomposition and the definition of rank-1-matrices fails.
A rank-3-matrix is given in the following manner: front slice identity, back slice just one non-zero element, the 1, so that it can not be annihilated by adding the first slice to it (for example $a_{000} = a_{011} = a_{101} = 1$ as only NZEs).
No higher rank can be attained: By Theorem 5.4 we can transform a matrix in the SR. The SR has at most five non-zero-elements. If it has five non-zero-elements, one can show that it has rank two or three (for details see below).

If it has four non-zero-elements or less we can always write the matrix as a sum of two rank-1-matrices or transform it to the rank three matrix mentioned above (also see below). □

The details of the proof above will be shown now.

We will now see what different outcomes we get from the SR, but let us first label the possible NZEs: $a_{001} = a, a_{010} = b, a_{100} = c, a_{101} = d, a_{111} = e$ (without loss of generality we can assume this constellation).

- First case: We have five NZEs (see Figure 5). If $bd^2 + 4ace \neq 0$, we can explicitly compute a decomposition as sum of 2 rank-1-matrices with Maple [6]. Therefore, the rank is 2 if $bd^2 + 4ace \neq 0$.
  If $bd^2 + 4ace = 0$ then the matrix can be transformed by 8 Gaussian steps (that do not change the rank) into a matrix of the form shown in Figure 5 with $a = c = d = e = 1$ and $b = -4$. By another 6 Gaussian steps this matrix can be transformed into (a permutation of) the independent matrix which has rank 3 by Proposition 5.8. Hence, if $bd^2 + 4ace = 0$ the rank is 3. As an anticipation we mention that the polynomial $bd^2 + 4ace$ is the so called hyperdeterminant of the matrix shown in Figure 5.

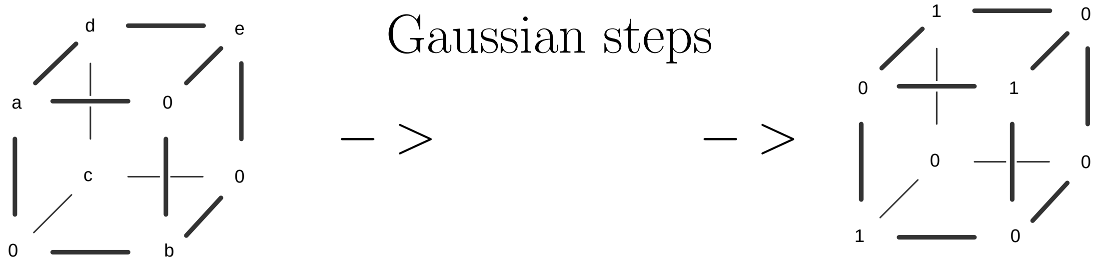


Figure 5: The case with 5 NZE (vanishing determinant)

- Second case: We have four NZEs. If either $a = 0$ (Figure 6), $c = 0$ (Figure 7) or $e = 0$ (analogue to case $a = 0$) , then a decomposition in two rank-1-matrices can be given (by definition these matrices which are written as a sum have not rank 1). Please note that the complex number x is the unique entry which makes the matrices containing it to rank-1-matrices. In the case $d = 0$, again a decomposition can be given (Figure 9): in particular we obtain

$$n_1 = \frac{\sqrt{abc}}{2\sqrt{e}}, \quad n_2 = \frac{a}{2}, \quad n_3 = \frac{b}{2}, \quad n_4 = \frac{\sqrt{abe}}{2\sqrt{c}},$$

$$n_5 = \frac{c}{2}, \quad n_6 = \frac{\sqrt{ace}}{2\sqrt{b}}, \quad n_7 = \frac{\sqrt{bce}}{2\sqrt{a}}, \quad n_8 = \frac{e}{2}$$

$$p_1 = \frac{-\sqrt{abc}}{2\sqrt{e}}, \quad p_2 = \frac{a}{2}, \quad p_3 = \frac{b}{2}, \quad p_4 = \frac{-\sqrt{abe}}{2\sqrt{c}},$$
$$p_5 = \frac{c}{2}, \quad p_6 = \frac{-\sqrt{ace}}{2\sqrt{b}}, \quad p_7 = \frac{-\sqrt{bce}}{2\sqrt{a}}, \quad p_8 = \frac{e}{2}$$

for the entries of the rank-1-matrices. For the case $b = 0$ we can use Gaussian steps to reduce it to an independent matrix.

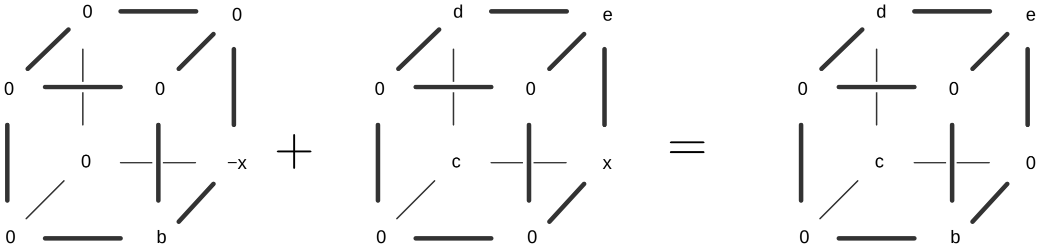


Figure 6: The case $a = 0$ (4 NZE)

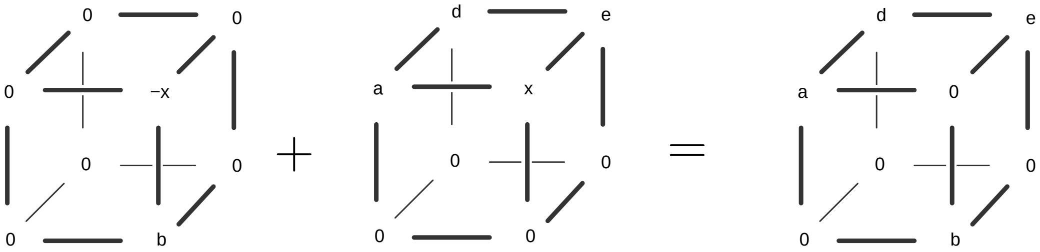


Figure 7: The case $c = 0$ (4 NZE)

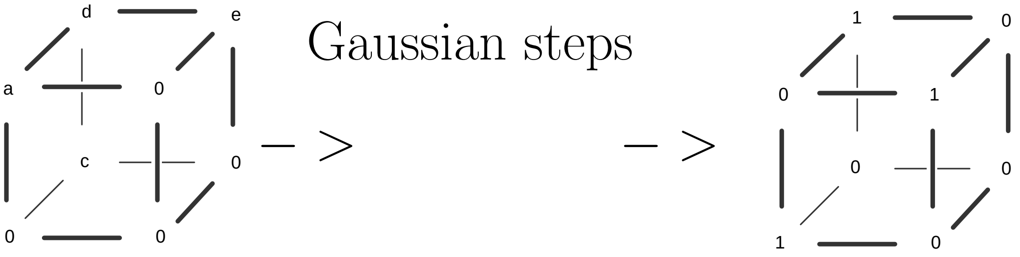


Figure 8: The case $b = 0$ (4 NZE)

On the other hand if $b = 0$ then the matrix can be transformed to the independent matrix and if $d = 0$ the outcome has rank two (analogous).

- Third case: Three NZEs. Only two possibilities occur in this case: either we have an independent matrix or the matrix can be classified as rank-2-matrix if you use Theorem 5.6. We do not give any more details since the occuring matrices are very easy to transform.

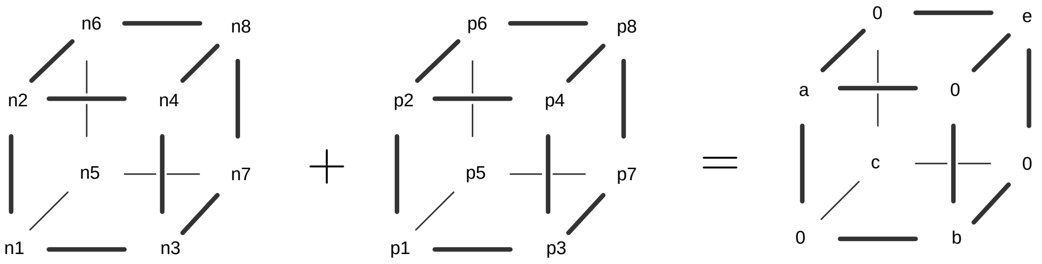


Figure 9: The case $d = 0$ (4 NZE)

- Fourth case: Less than three NZEs. By the examples outlined above and the definition of rank, the rank of these matrices is obvious.

By this disjoint deduction of the SR we can classify all types of $2 \times 2 \times 2$ hypermatrices. This can also be seen in Figure 10 (note that the tree is not collectively exhaustive since we left out cubes which are symmetric to others). Mentionable is the fact that an independent matrix occurs in every case in which the rank is three.

An interesting question which arises immediately is whether there exists a connection between rank and the vanishing of a determinant like in the two-dimensional case. Please note that the following statement excludes rank-3-matrices because we need more structural knowledge for them. Further note that the GKZ-hyperdeterminant (see [3]) of a $2 \times 2 \times 2$ hypermatrix $A = (a_{jkl})$, $j, k, l \in \{0, 1\}$ is given by:

$$Det(A) = (a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{011}^2 a_{100}^2)$$

$$-2(a_{000}a_{001}a_{110}a_{111} + a_{000}a_{010}a_{101}a_{111} + a_{000}a_{011}a_{100}a_{111} + a_{001}a_{010}a_{101}a_{110}$$

$$+a_{001}a_{011}a_{110}a_{100} + a_{010}a_{011}a_{101}a_{100}) + 4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111})$$

Theorem 5.10 *The GKZ-hyperdeterminant of all* $2 \times 2 \times 2$ *hypermatrices* $A = (a_{jkl})$, $j, k, l \in \{0, 1\}$ *with complex entries and rank smaller or equal to two vanishes except for most matrices of rank two.*

Proof. For rank-0-matrices the theorem is trivial. The same applies for rank-1-matrices since the hyperdeterminant is invariant under Gaussian steps (see [3]).
Now we first give an example of a rank-2-matrix whose hyperdeterminant does not vanish and then of one whose hyperdeterminant does vanish. Looking at the matrix which has only two non-zero elements in the diagonal of the cube (for example $a_{000} = a_{111} = 1$), one can easily check that all terms of the

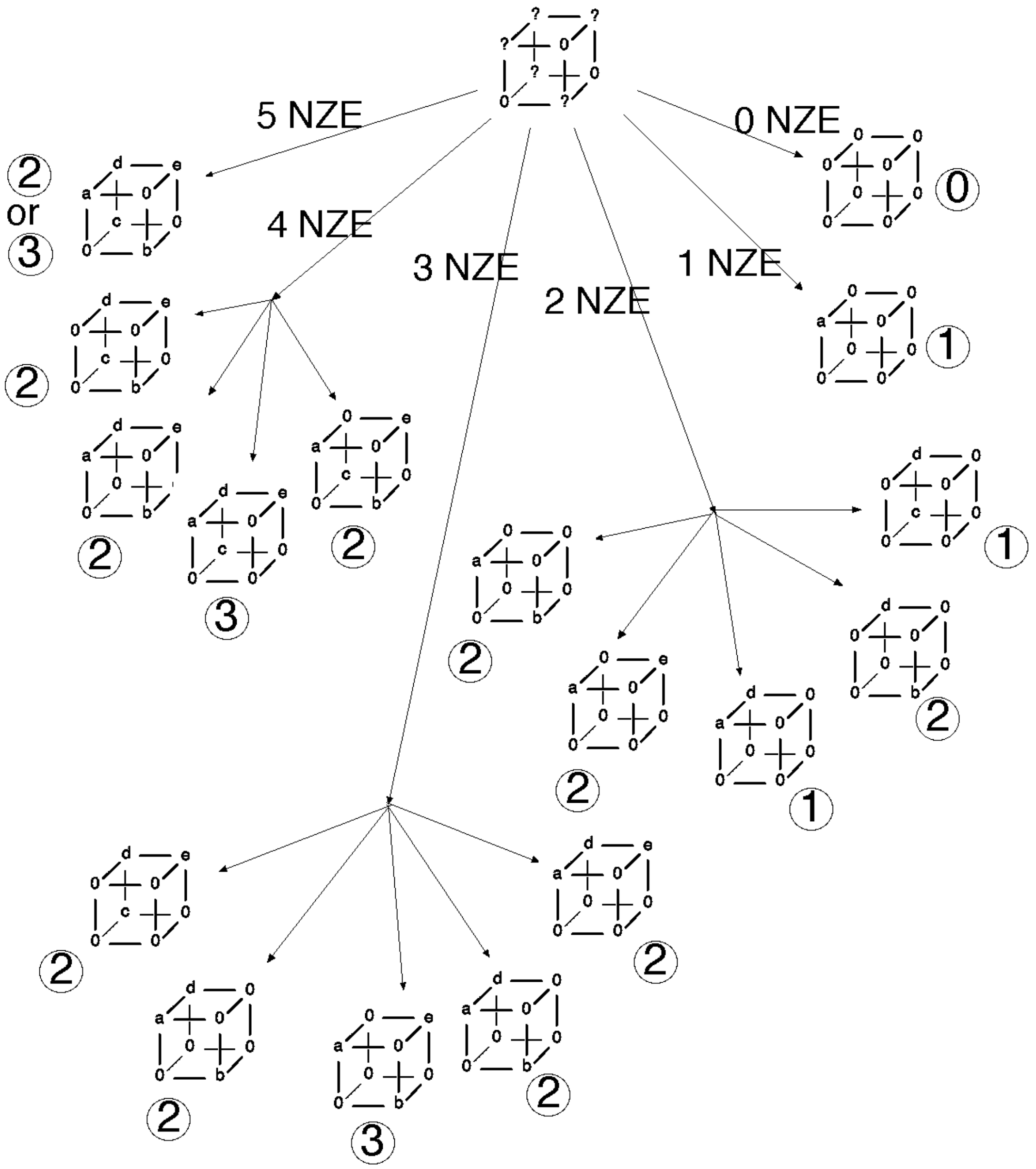


Figure 10: The Algorithmic Tree (the number in the circles is the rank)

hyperdeterminant except the one concerning this diagonal vanish.
On the other hand look at the matrix which has the 2×2 identity-matrix in one slice and zeros in the other (for example $a_{000} = a_{011} = 1$), then all terms of the hyperdeterminant vanish.
For most matrices of rank two the determinant does not vanish since the determinant is a homogeneous polynomial. In the generic case we will not find the zero set of the polynomial. □

We have seen that the existence of NZEs in certain positions of the cube is crucial for the rank of a hypermatrix. We would like to specify some fundamental observations.

Lemma 5.11 *By changing a non-zero-element to zero (or vice versa) in a hypermatrix of arbitrary spatial dimension with complex entries, the rank stays the same, increases by one or decreases by one.*

Proof. Assume that the rank does change. By the definition of rank we see that if a hypermatrix has rank r (with r a positive integer) and is modified in the way described above, an upper bound is given by the canonical decomposition in $r + 1$ rank-1-matrices.
On the other hand assume that the the rank of the modified matrix would be lesser than r. Without loss of generality the new matrix has rank $r - 2$. By using the definition of rank and the fact that the modification amounts to adding a rank-1-matrix, a contradiction is created. $\square$

Corollary 5.12 *The rank of a hypermatrix of arbitrary spatial dimension stays the same or changes at most by one if the hypermatrix is modified in a way which amounts to the addition of a rank-1-matrix.*

Proof. Obvious. $\square$

5.2 The Orbits in the $2 \times 2 \times 2$ Case under the $GL_2(\mathbb{C})^3$-action

Being inspired by the physical interpretation, we now examine how the so called local operations, which act on the matrix, induce certain orbits.

Theorem 5.13 *Local operations on the set of all* $2 \times 2 \times 2$ *hypermatrices* $A = (a_{jkl})$, $j, k, l \in \{0, 1\}$ *have seven orbits: four orbits in rank 2 and one orbit in each of the other ranks.*

Proof. It is obvious that for rank-0-matrices there is just one orbit: the rank-0-matrix itself.

For rank-1-matrices there is also just one orbit. To see that, we remember that every rank-1-matrix can be reduced to a single non-zero-element by using proper directions in the three different spatial dimensions. This non-zero-element can be translated to any position in the cube by Gaussian steps. Additionally, it can be scaled to any non-zero-element.

This implies that any rank-1-matrix, let us call it B, can be reproduced by locating the non-zero-element in a position in which B has a non-zero-element and by creating non-zero-elements on positions in which B has non-zero-elements. The place which was initially chosen for the non-zero-element does not matter, since by definition of a rank-1-matrix every other position of non-zero-elements can be reached by Gaussian steps.

Now let us look at the rank-3-matrices before we proceed to examine the interesting case of rank 2. Here we use a similar argument like in the proof above. By the possibilities which can be deduced from the SR, all which lead to a rank-3-matrix lead to the independent matrix. Thus the independent matrix occurs in every case of a rank-3-matrix. The only issue we have to be worried about is whether it matters how the NZEs are set in the cube. But it turns out that by a few Gaussian steps or simply by switching the slices, we can nontrivially permutate one direction. So every possible configuration of the independent matrix can be reached since the independent matrix is symmetric in the three spatial dimensions. We see that there is just one orbit.

Finally we have to inspect rank 2: First keep in mind that any rank-2-matrix $A = (a_{jkl})$, $j, k, l \in \{0, 1\}$ can be reduced to just two non-zero-elements (NZE). Without loss of generality, one NZE is a_{000} (this is possible because we are just interested how the other NZE is located with respect to the first NZE, in other words: like in the case of the independent matrix it is possible to mirror the NZEs on planes which dissect the cube in two slices by simple Gaussian steps). For the other NZE, we have exactly four possibilities (otherwise A would have rank 1), so the other NZE can either be a_{111}, a_{011}, a_{110} or a_{101}. Let us call the

corresponding matrices $A1$, $A2$, $A3$ and $A4$.
Now we realize that the matrix which has a_{111} as NZE can not be in the same orbit as the other three possible matrices since its hyperdeterminant does not vanish in contrary to those of the others. This uses the fact that the hyperdeterminant is invariant under Gaussian steps (see [3]).
So we just have to look at the remaining three posibilities: We can say that $A2$ is not in the orbit(s) of $A3$ and $A4$ because to transform $A2$ either in $A3$ or $A4$, we have to create a NZE in the slice $\{a_{1kl}; k, l \in 0, 1\}$. By attempting to do this we necessarily have to add the front slice $\{a_{0kl}; k, l \in 0, 1\}$ to the back slice. But because of the fact that the back slice $\{a_{1kl}; k, l \in 0, 1\}$ had only zero-elements (ZE), the front slice $\{a_{0kl}; k, l \in 0, 1\}$ is "copied" and no matter which Gaussian steps are performed, the back slice remains a scalar multiple of the front slice. So $A2$ can never be transformed in $A3$ or $A4$ by Gaussian steps.
Finally we have to examine $A3$ and $A4$. But here the same argument applies by regarding the slices $\{a_{j0l}; j, l \in 0, 1\}$ and $\{a_{j1l}; j, l \in 0, 1\}$. $\square$

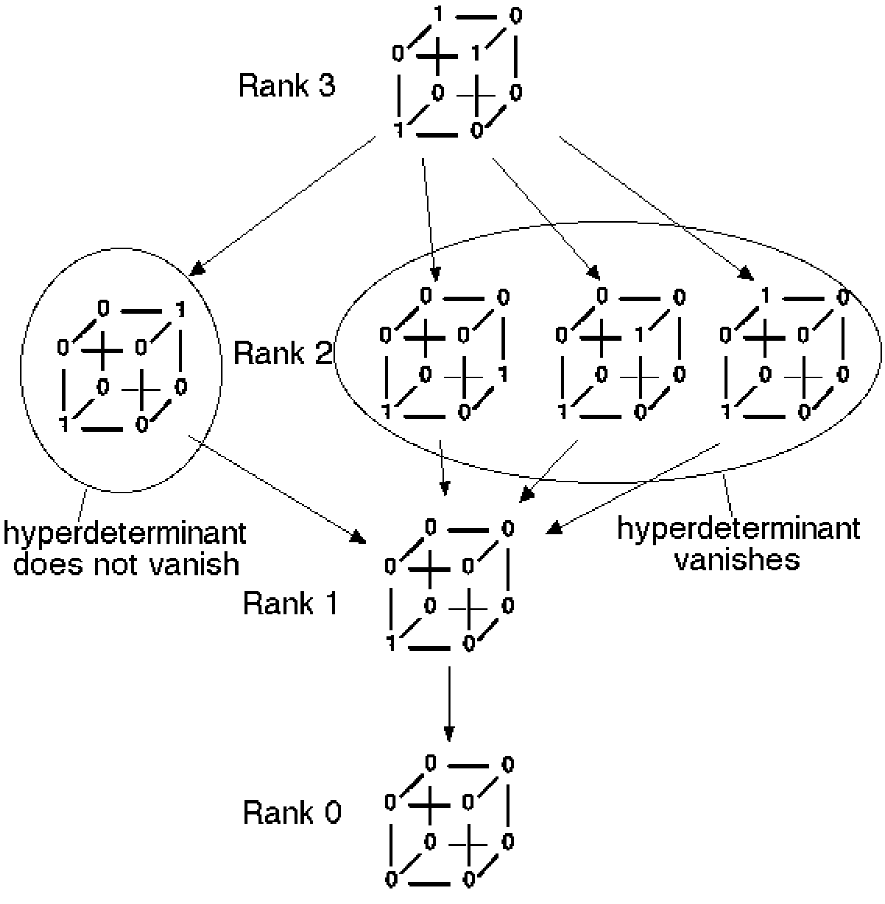


Figure 11: The Orbit Tree

It is sometimes helpful to have representants of the orbits.

Corollary 5.14 *The following matrices* $A = (a_{000}, a_{010}, a_{001}, a_{011}, a_{100}, a_{110}, a_{101}, a_{111})$ *are representants of the seven different orbits:* $(0,0,0,0,0,0,0,0)$ *(rank 0),* $(1,0,0,0,0,0,0,0)$ *(rank 1),* $(1,0,0,1,0,0,0,0)$, $(1,0,0,0,0,0,0,1)$, $(1,0,0,0,0,0,1,0)$, $(1,0,0,0,0,1,0,0)$*(rank 2) and* $(1,0,0,1,0,0,1,0)$ *(rank 3).*

Proof. Follows immediately from the proof above. □

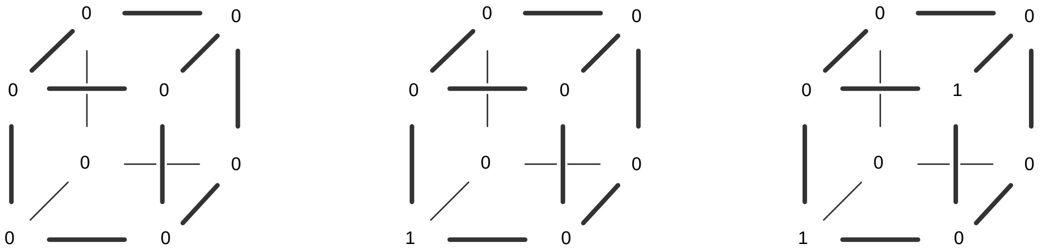


Figure 12: Representants of rank 0, 1 and 2 orbits

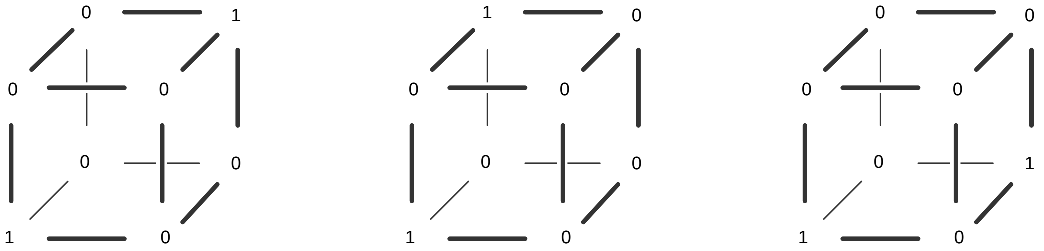


Figure 13: Representants of rank 2 orbits

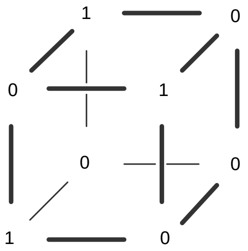


Figure 14: Representant of the rank 3 orbit

With the gained structural understanding, we can complete statements given above also for rank-3-matrices.

Corollary 5.15 *Every rank-3-hypermatrix* $A = (a_{jkl})$, $j, k, l \in \{0, 1\}$ *with complex entries can be transformed by Gaussian steps in the* independent matrix*. Especially every rank-3-matrix can be transformed by Gaussian steps in a matrix with only three NZE.*

Proof. Trivial since rank-3-matrices have just one orbit under local operations. □

Corollary 5.16 *The GKZ-hyperdeterminant of all rank-3-matrices* $A = (a_{jkl})$, $j, k, l \in \{0, 1\}$ *vanishes.*

Proof. Since the rank-3-matrices have only one orbit and the GKZ-hyperdeterminant of the independent matrix vanishes, the corollary is proved. □

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