COMPUTING NODE POLYNOMIALS FOR PLANE CURVES

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ABSTRACT. Algebraic plane degree d curves with δ nodes are enumerated by node polynomials $N_\delta(d)$, if d is large enough. Vainsencher and Kleiman–Piene computed them up to 6 and 8 nodes, respectively. We compute $N_\delta(d)$ for $\delta \leq 13$ using the combinatorics of labeled floor diagrams via an explicit algorithm that has no constrain on the number of nodes.

Furthermore, we improve the threshold for polynomiality and verify Göttsche's conjecture on the optimal threshold up to $\delta \leq 13$. We also compute the first 8 coefficients of $N_\delta(d)$, for general δ, settling and extending a conjecture of Di Francesco and Itzykson from 1994.

This is a preliminary draft. Suggestions welcome.

1. INTRODUCTION

Counting algebraic plane curves is a very old problem. In 1848 Jakob Steiner computed the number of such curves of degree d with 1 node through $\frac{d(d+3)}{2} - 1$ generic points in $\mathbb{P}^2$ to be $3(d-1)^2$. Much effort has been put forth since towards answering the following question.

How many (possibly reducible) nodal plane degree d curves with δ nodes pass through $\frac{d(d+3)}{2} - \delta$ generic points?

In 1994 Di Francesco and Itzykson [4] conjectured this number, the *Severi degree* $N^{d,\delta}$, to be given by *node polynomials* $N_\delta(d)$ in the degree d, for a fixed number of nodes δ, if d is large enough. The number of point conditions, $\frac{d(d+3)}{2} - \delta$, is such that the the curve count is finite. In [7] Göttsche conjectured an explicit description of these polynomials for projective algebraic surfaces.

In the 19th century $N_\delta(d)$ was known for $\delta \leq 3$. The node polynomials for $\delta \leq 6$ were computed by Vainsencher [10] in 1995 and for $\delta \leq 8$ by Kleiman–Pienne [8] in 2001. Recently, Fomin and Mikhalkin established polynomiality for arbitraty δ in [6] where they showed that $N^{d,\delta} = N_\delta(d)$ for $d \geq 2\delta$ with methods from tropical geometry and floor decompositions. Using their techniques we design an explicit algorithm for arbitrary δ and compute all node polynomials $N_\delta(d)$ for $\delta \leq 13$ reproducing earlier results for $\delta \leq 8$.

Date: November 18, 2009.

Theorem 1.1. *The node polynomials $N_\delta(d)$, for $9 \leq \delta \leq 13$, are as listed in Appendix A.*

A complete list of all $N_\delta(d)$ for $\delta \leq 13$ is implicitely given in Theorem 3.1 using generating functions. By refining Fomin's and Mikhalkin's argument in [6], their threshold for polynomiality can be reduced.

Theorem 1.2. *For $\delta \geq 0$ and $d \geq \delta$ it holds that $N_\delta(d) = N^{d,\delta}$.*

In [7] Göttsche conjectured the threshold value for polynomiality of $N^{d,\delta}$ to be $d_0(\delta) = \lceil \frac{\delta}{2} \rceil + 1$. This was varified for $\delta \leq 8$ by Kleiman and Piene in [8]. By direct computation we can push it further.

Theorem 1.3. *Göttsche's threshold value is correct and sharp for $\delta \leq 13$.*

Di Francesco and Itzykson [4] suggested a threshold of $d_0(d) = \left\lceil \frac{3}{2} + \sqrt{2\delta + \frac{1}{4}} \right\rceil$ in [4] (so that $d \geq d_0(\delta)$ is equivalent to $\delta \leq \frac{(d-1)(d-2)}{2}$). However, this is false.

Proposition 1.4. *For $\delta = 13$ and $d = 7$:*

$$N^{7,13} = 131024290671 \neq 130587800463 = N_{13}(7)$$

despite $13 \leq \frac{6 \cdot 5}{2}$.

The last two results are a direct application of Algorithm 1. In [4] Di Francesco and Itzykson also conjectured the first seven terms of the node polynomial $N_\delta(d)$, for arbitrary δ. We can affirm and extend their assertion.

Theorem 1.5. *The first eight coefficients of $N_\delta(d)$ are given by*

$$\begin{aligned}
N_\delta(d) = \frac{3^\delta}{\delta!} \Bigg[& d^{2\delta} - 2\delta d^{2\delta-1} - \frac{\delta(\delta-4)}{3} d^{2\delta-2} + \frac{\delta(\delta-1)(20\delta-13)}{6} d^{2\delta-3} + \\
& - \frac{\delta(\delta-1)(69\delta^2-85\delta+92)}{54} d^{2\delta-4} - \frac{\delta(\delta-1)(\delta-2)(702\delta^2-629\delta-286)}{270} d^{2\delta-5} + \\
& \frac{\delta(\delta-1)(\delta-2)(6028\delta^3-15476\delta^2+11701\delta+4425)}{3240} d^{2\delta-6} + \\
& \frac{\delta(\delta-1)(\delta-2)(\delta-3)(13628\delta^3-6089\delta^2-29572\delta-24485)}{11340} d^{2\delta-7} + \cdots \Bigg]
\end{aligned}$$

The proofs of Theorems 1.1 and 1.5 are algorithmic in nature and involve a computer computation. We describe both algorithms in Sections 3 and 5, respectively. The first algorithm computes node polynomials $N_\delta(d)$ for arbitrary δ (given enough CPU time), the second a prescribed number of leading terms of $N_\delta(d)$ *if the algorithm terminates*. This always happens conjecturally but a proof is still missing.

A plane degree d curve with δ nodes must be irreducible by Bézout's Theorem if $d \geq \delta + 2$. In this case $N^{d,\delta} = N_{d,g}$, where $N_{d,g}$ is the Gromow-Witten invariant enumerating irreducible plane curves of degree d and genus

g through $3d+g-1$ generic points, where $g+\delta = \frac{(d-1)(d-2)}{2}$. With this relation in mind we refer to δ as the *cogenus* of a curve also. Algorithm 1 (with minor adjustments) can compute $N_{d,g}$ directly, without a recursion that involves smaller (and relative) Gromov-Witten invariants like in Caporaso–Harris' famous recursion [3].

Labeled floor diagrams (which we will define in Section 2) were introduced by Brugallé and Mikhalkin in [1] and [2]. These combinatorial gadgets, if counted with the right multiplicity (i.e., their number of "markings", see also Section 2), are in bijection with algebraic plane curves with certain presribed properties. There is a dictionary between properties of algebraic curves and labeled floor diagrams that respects this bijection. For example, the degree, the number of nodes and the genus (if the curve is irreducible) of a curve can be read of from its corresponding labeled floor diagram. In this sense labeled floor diagrams are the right combinatorial objects to enumerate nodal plane curves. The idea is to "get rid of all geometry" hence reducing curve enumeration problems to purely combinatorial questions.

In principle, once polynomiality of the Severi degrees $N^{d,\delta}$ is established with some threshold value, one could use the Caporaso-Harris recursion [3] to compute all node polynomials. Their recursion computes all Severi degrees which can be used to compute all $N_\delta(d)$ by polynomial interpolation. However, this method is less feasible as the Caporaso-Harris recursion depends on relative Severi degrees also and is slower than our more direct approach.

This paper is organized as follows. In Section 2 we review labeled floor diagrams, their markings and how they enumerate plane algebraic curves. The algorithm that computes the new node polynomials in Theorem 1.1 is described in detail in Section 3. Theorem 1.2 is proved in Section 4. In Section 5 we prove Theorem 1.5 by giving an explicit algorithm.

I thank Sergey Fomin for suggesting this problem and fruitful guidance. I also thank Erwan Brugallé and Gregory Mikhalkin for valuable comments and suggestions.

2. Labeled Floor Diagrams

Labeled floor diagrams are the combinatorial gadgets which, if counted correctly, enumerate nodal plane curves. They play a similar role as partitions in Schubert Calculus, representation theory or the theory of symmetric functions. Brugallé and Mikhalkin introduced them in [1] (in slightly different notation) and studied them further in [2]. To keep this paper self-contained and to fix notation we review them here following [6].

Definition 2.1. A *labeled floor diagram* $\mathcal{D}$ on a vertex set $\{1, \ldots, d\}$, where $1 < \cdots < d$, is a directed graph with possibly multiple edges and positive integer edge weights $w(e) = w(i \xrightarrow{e} j)$ satisfying:

(1) The edge directions respect the order of the vertices, i.e., for each edge $i \xrightarrow{e} j$ of $\mathcal{D}$ we have $i < j$.

(2) (Divergence Condition) For each vertex j of $\mathcal{D}$:

$$\operatorname{div}(j) \stackrel{\text{def}}{=} \sum_{\substack{\text{edges } e \\ j \xrightarrow{e} k}} w(e) - \sum_{\substack{\text{edges } e \\ i \xrightarrow{e} j}} w(e) \leq 1.$$

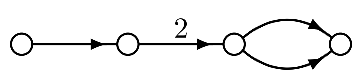
This means that at every vertex of $\mathcal{D}$ the number of outgoing edges (counted with their weights) is larger by at most 1 than the number of incoming edges (counted with their weights).

Note that we do not require $\mathcal{D}$ to be connected (which is required in [6]). The *degree* of a labeled floor diagram is the number of its vertices. A labeled floor diagram $\mathcal{D}$ is *connected* if its underlying graph is. Its *genus* is the genus of the underlying graph (or the first Betti number of the underlying topological space) if $\mathcal{D}$ is connected. The *cogenus* of a connected labeled floor diagram $\mathcal{D}$ of degree d and genus g is given by $\delta(\mathcal{D}) = \frac{(d-1)(d-2)}{2} - g$. If $\mathcal{D}$ is not connected let $d_1, d_2, \ldots$ and $\delta_1, \delta_2, \ldots$ be the degrees and the cogenera, respectively, of its connected components. The *cogenus* of $\mathcal{D}$ is $\sum_j \delta_j + \sum_{j<j'} d_j d_{j'}$. Via the correspondence between labeled floor diagrams and algebraic curves (Theorem 2.5 in [2]) these notions correspond literally to the respective analogues for algebraic curves. Connectedness corresponds to irreducibility of algebraic curves. Lastly, a marked floor diagram $\mathcal{D}$ has *multiplicity*[1]

$$\mu(\mathcal{D}) = \sum_{\text{edges } e} w(e)^2.$$

We draw labeled floor diagrams using the convention that vertices in increasing order are arranged left to right. Edge weights of 1 are omitted.

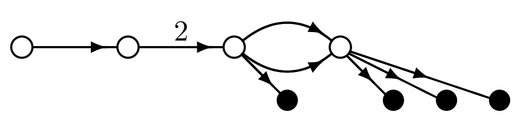
Example 2.2. An example of a labeled floor diagram with degree $d = 4$, genus $g = 1$, cogenus $\delta = 2$, divergences $1, 1, 0, -2$, and multiplicity $\mu = 4$ is drawn below.



To enumerate algebraic curves via labeled floor diagrams we need the notion of multiplicity of such diagrams which is slightly technical.

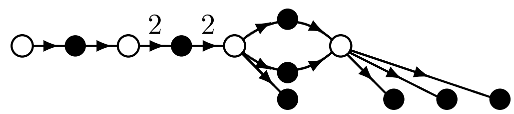
Definition 2.3. A *marking* of a labeled floor diagram $\mathcal{D}$ is defined by the following three step process that we illustrate in the case of Example 2.2.

Step 1: For each vertex j of $\mathcal{D}$ create $1 - div(j)$ many new vertices and connect them to j with new edges directed away from j.

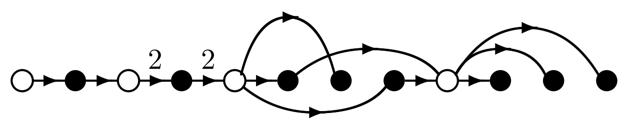


[1]If floor diagrams are viewed as floor contractions of tropical plane curves this corresponds to the notion of multiplicity of tropical plane curves.

Step 2: Subdivide each edge of the original labeled floor diagram $\mathcal{D}$ into two directed edges by introducing a new vertex for each edge. The new edges inherit their weights and orientations. Call the resulting graph $\tilde{\mathcal{D}}$.



Step 3: Linearly order the vertices of $\tilde{\mathcal{D}}$ extending the order of the vertices of the original labeled floor diagram $\mathcal{D}$.



The extended graph $\tilde{\mathcal{D}}$ together with the linear order on its vertices is called a *marked floor diagram*, or a *marking* of the original labeled floor diagram $\mathcal{D}$.

We want to count marked floor diagrams up to equivalence. Two marked floor diagrams $\tilde{\mathcal{D}}_1$, $\tilde{\mathcal{D}}_2$ are *equivalent* if $\tilde{\mathcal{D}}_1$ can be obtained from $\tilde{\mathcal{D}}_2$ by permuting edges without changing their weights, i.e., if there exists a vertex preserving automorphism of weighted graphs mapping $\tilde{\mathcal{D}}_1$ to $\tilde{\mathcal{D}}_2$. The *number of markings* $\nu(\mathcal{D})$ is the number of marked floor diagrams $\tilde{\mathcal{D}}$ up to equivalence.

Example 2.4. The labeled floor diagram $\mathcal{D}$ of Example 2.2 has $\nu(\mathcal{D}) = 7$: The extra 1-valent vertex connected to the third white vertex from the left can be inserted in three ways between the third and fourth white vertex (up to equivalence) and in four ways right to the fourth white vertex (again up to equivalence).

Now we can make precise how to rephrase the initial question of this paper in terms of combinatorics of labeled floor diagrams.

Theorem 2.5 (Theorem 3.6 of [1])**.**

(1) *The Severi degree $N^{d,\delta}$, i.e., the number of (possibly reducible) nodal curves in $\mathbb{P}^2$ of degree d with δ nodes through $\frac{d(d+3)}{2} - \delta$ generic points, is equal to*

$$N^{d,\delta} = \sum_{\mathcal{D}} \mu(\mathcal{D})\nu(\mathcal{D}),$$

where $\mathcal{D}$ runs over all (possibly disconnected) labeled floor diagrams of degree d and cogenus δ.

(2) *The Gromov-Witten invariant $N_{d,g}$, i.e., the number of irreducible curves in $\mathbb{P}^2$ of degree d and genus g through $3d+g-1$ generic points, is equal to*

$$N_{d,g} = \sum_{\mathcal{D}} \mu(\mathcal{D})\nu(\mathcal{D}),$$

where $\mathcal{D}$ runs over all connected *labeled floor diagrams of degree d and genus g.*

3. Computing node polynomials

In this section we give an explicit algorithm that symbolically computes the node polynomial $N_\delta(d)$, for given $\delta \geq 0$. While setting up notation we mostly follow Section 5 of [6]. Our goal is to prove Theorem 1.1, which we rephrase here in more compact notation.

Theorem 3.1. *The node polynomials $N_\delta(d)$, for $\delta \leq 13$, are given by the generating function $\sum_{\delta\geq 0} N_\delta(d)x^\delta$ via the transformation*

$$\sum_{\delta\geq 0} N_\delta(d)x^\delta = \exp\left(\sum_{\delta\geq 0} Q_\delta(d)x^\delta\right),$$

where

$$\begin{aligned}
Q_0(d) &= 1,\\
Q_1(d) &= 3(d-1)^2,\\
Q_2(d) &= \tfrac{-3}{2}(d-1)(14d-25),\\
Q_3(d) &= \tfrac{1}{3}(690d^2 - 2364d + 1899),\\
Q_4(d) &= \tfrac{1}{4}(-12060d^2 + 47835d - 45207),\\
Q_5(d) &= \tfrac{1}{5}(217728d^2 - 965646d + 1031823),\\
Q_6(d) &= \tfrac{1}{6}(-4010328d^2 + 19451628d - 22907925),\\
Q_7(d) &= \tfrac{1}{7}(74884932d^2 - 391230216d + 499072374),\\
Q_8(d) &= \tfrac{1}{8}(-1412380980d^2 + 7860785643d - 10727554959),\\
Q_9(d) &= \tfrac{1}{9}(26842726680d^2 - 157836614730d + 228307435911),\\
Q_{10}(d) &= \tfrac{1}{10}(-513240952752d^2 + 3167809665372d - 4822190211285),\\
Q_{11}(d) &= \tfrac{1}{11}(9861407170992d^2 - 63560584231524d + 101248067530602),\\
Q_{12}(d) &= \tfrac{1}{12}(-190244562607008d^2 + 1275088266948600d - 2115732543025293),\\
Q_{13}(d) &= \tfrac{1}{13}(3682665360521280d^2 - 25576895657724768d + 44039919476860362).
\end{aligned}$$

In particular, all $Q_\delta(d)$, for $1 \leq \delta \leq 13$, are quadratic.

The quadraticity of the $Q_\delta(d)$'s is quite remarkable as the node polynomials $N_\delta(d)$ are pretty complicated polynomials (see Appendix A) of degree 2δ. Göttsche conjectured in [7] that all $Q_\delta(d)$ are quadratic. This theorem proves his conjecture for $\delta \leq 13$.

The basic idea of the algorithm is to decompose floor diagrams into smaller bulding blocks. These gadgets will be crucial in the proofs of all theorems in this paper.

Definition 3.2. A *template* Γ is a directed graph on vertices $\{0, \ldots, l\}$, where $0 < \cdots < l$ and $l \geq 1$, with possibly multiple edges and edge weights $w(e) \in \mathbb{Z}_{>0}$, satisfying:

(1) If $i \xrightarrow{e} j$ is an edge then $i < j$.
(2) Every edge $i \xrightarrow{e} i+1$ has weight $w(e) \geq 2$. (No "short edges.")
(3) For each vertex j, $1 \leq j \leq l-1$, there is an edge "covering" it, i.e., there exists an edge $i \xrightarrow{e} k$ with $i < j < k$.

The last condition says that templates cannot be split at vertices into smaller templates. Every template comes with a list of numerical data associated to it. Its *length* $l(\Gamma)$ is the number of vertices minus 1. The product of squares of its edge weights is its *multiplicity* $\mu(\Gamma)$. Its cogenus $\delta(\Gamma)$ is

$$\delta(\Gamma) = \sum_{i \xrightarrow{e} j} \Big[(j-i)w(e) - 1\Big].$$

For $1 \leq j \leq l(\Gamma)$ let $\varkappa_j = \varkappa_j(\Gamma)$ denote the sum of the weights of edges $i \xrightarrow{e} k$ with $i < j \leq k$. Set $\varkappa(\Gamma) = (\varkappa_1, \ldots, \varkappa_l)$ and define

$$k_{min}(\Gamma) = \max_{1 \leq j \leq l} (\varkappa_j - j + 1).$$

This makes $k_{min}(\Gamma)$ the smallest positive integer k such that Γ can appear in a floor diagram on $\{1, 2, \ldots\}$ with left-most vertex k. Lastly, set

$$\varepsilon(\Gamma) = \begin{cases} 1 & \text{if all edges arriving at } l \text{ have weight } 1, \\ 0 & \text{otherwise.} \end{cases}$$

Figure 1 (Figure 10 taken from [6]) lists all templates Γ with $\delta(\Gamma) \leq 2$.

A labeled floor diagram $\mathcal{D}$ with d vertices decomposes into an ordered collection $(\Gamma_1, \ldots, \Gamma_m)$ of templates as follows. Add an addition vertex $d+1$ $(> d)$ to $\mathcal{D}$ along with, for every vertex j of $\mathcal{D}$, $1 - div(j)$ new edges of weight 1 connecting j with the new vertex $d+1$. The resulting floor diagram $\mathcal{D}'$ has divergence 1 at every vertex coming from $\mathcal{D}$. Now remove all *short edges* from $\mathcal{D}'$, that is all edges of weight 1 between consecutive vertices. The result is an ordered collection of templates $(\Gamma_1, \ldots, \Gamma_m)$, listed left to right, and it is not hard to see that $\sum \delta(\Gamma_i) = \delta(\mathcal{D})$. This process is reversible once we record the smallest vertex k_i of each templates Γ_i (see Example 3.3).

Example 3.3. An example of the decomposition of a labeled floor diagram into templates is illustrated below. Here, $k_1 = 2$ and $k_2 = 4$.

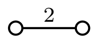
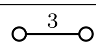
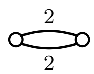
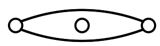
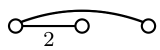
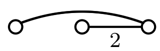
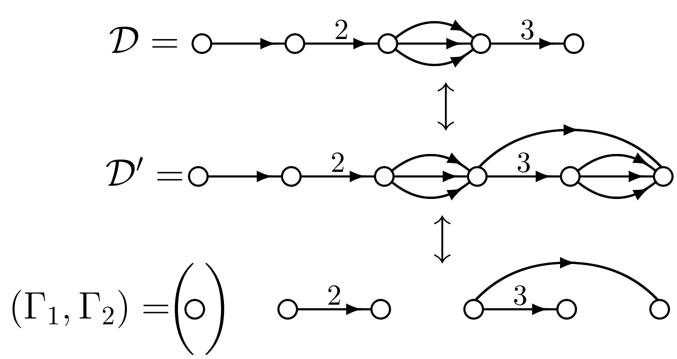
Γ	$\delta(\Gamma)$	$\ell(\Gamma)$	$\mu(\Gamma)$	$\varepsilon(\Gamma)$	$\varkappa(\Gamma)$	$k_{\min}(\Gamma)$	$P(\Gamma,k)$
	1	1	4	0	(2)	2	$k-1$
	1	2	1	1	(1,1)	1	$2k+1$
	2	1	9	0	(3)	3	$k-2$
	2	1	16	0	(4)	4	$\binom{k-2}{2}$
	2	2	1	1	(2,2)	2	$\binom{2k}{2}$
	2	2	4	1	(3,1)	3	$2k(k-2)$
	2	2	4	0	(1,3)	2	$2k(k-1)$
	2	3	1	1	(1,1,1)	1	$3(k+1)$
	2	3	1	1	(1,2,1)	1	$k(4k+5)$

FIGURE 1. Templates with $\delta(\Gamma) \leq 2$.



To each template Γ we associate a polynomial that records the number of "markings" of Γ: For $k \in \mathbb{Z}_{>0}$ let $\Gamma_{(k)}$ denote the graph obtained from Γ by first adding $k+i-1-\varkappa_j$ short edges connecting $i-1$ to i, for $1 \leq i \leq l(\Gamma)$, and then subdividing each edge of the resulting graph by introducing one new vertex for each edge. By Lemma 5.6 of [6] the number of linear extentions (up to equivalence) of the vertex poset of the new graph $\Gamma_{(k)}$ extending the vertex order of Γ is a polynomial in k, if $k \geq k_{min}(\Gamma)$, which we denote with $P(\Gamma, k)$ (see Figure 1). The number of markings of a labeled floor diagram $\mathcal{D}$ decomposing into templates $(\Gamma_1, \cdots, \Gamma_m)$ is then

$$\nu(\mathcal{D}) = \prod_{i=1}^{m} P(\Gamma_i, k_i),$$

where k_i is the smallest vertex of Γ_i in $\mathcal{D}$. The algorithm is based on

Theorem 3.4 ([6], Proof of Theorem 5.1)**.** *The Severi degree $N^{d,\delta}$, for $d, \delta \geq 0$, is given by the following template decomposition formula*

$$\sum_{\Gamma_1,\dots,\Gamma_m} \prod_{i=1}^{m} \mu(\Gamma_i) \sum_{k_m = k_{min}(\Gamma_m)}^{d-l(\Gamma_m)+\varepsilon(\Gamma_m)} P(\Gamma_m, k_m) \cdots \sum_{k_1 = k_{min}(\Gamma_1)}^{k_2 - l(\Gamma_1)} P(\Gamma_1, k_1). \tag{3.1}$$

where the first sum is over all ordered collections of templates $(\Gamma_1, \dots, \Gamma_m)$, for all $m \geq 1$, with $\sum_{i=1}^{m} \delta(\Gamma_i) = \delta$, and the sums indexed by k_i, for $1 \leq i < m$, are over $k_{min}(\Gamma_i) \leq k_i \leq k_{i+1} - l(\Gamma_i)$.

Expression (3.1) can be evaluated symbollically, using the following two lemmata. The first is Faulhaber's formula [5] from 1631 for discrete integration of polynomials. The second treats lower limits of interated discrete integrals and its proof is straightforward. Here B_j denotes the jth Bernoulli number with the convention that $B_1 = +\frac{1}{2}$.

Lemma 3.5. *Let $f(k) = \sum_{i=0}^{d} c_i k^i$. Then, for $n \geq 0$,*

$$F(n) \stackrel{def}{=} \sum_{k=0}^{n} f(k) = \sum_{s=0}^{d} \frac{c_s}{s+1} \sum_{j=0}^{s} \binom{s+1}{j} B_j n^{s+1-j}. \tag{3.2}$$

In particular, $deg(F) = deg(f) + 1$.

Lemma 3.6. *Let $f(k_1)$ and $g(k_2)$ be polynomials in k_1 and k_2, respectively, and let $a_1, b_1, a_2, b_2 \in \mathbb{Z}_{\geq 0}$. Furthermore, let $F(k_2) = \sum_{k_1=a_1}^{k_2-b_1} f(k_1)$ be a discrete anti-derivative of $f(k_1)$, where $k_2 \geq a_1 + b_1$. Then, for $n \geq \max(a_1 + b_1 + b_2, a_2 + b_2)$,*

$$\sum_{k_2=a_2}^{n-b_2} g(k_2) \sum_{k_1=a_1}^{k_2-b_1} f(k_1) = \sum_{k_2=\max(a_1+b_1, a_2)}^{n-b_2} g(k_2) F(k_2).$$

Example 3.7. An illustration of Lemma 3.6 is the following iterated discrete integral:

$$\sum_{k_2=1}^{n} \sum_{k_1=1}^{k_2-1} 1 = \sum_{k_2=1}^{n} \left\{ \begin{array}{ll} k_2 - 1 & \text{if } k_2 \geq 2 \\ 0 & \text{if } k_2 = 1 \end{array} \right\} = \sum_{k_2=2}^{n} \big(k_2 - 1\big).$$

This shows that when such expressions are evaluated symbollically we sometimes need to change lower limits.

Now we can state Algorithm 1 which computes node polynomials $N_\delta(d)$ for an arbitrary number of nodes δ. The first step, the template enumeration, is explained later in the section.

Data: The cogenus δ.
Result: The node polynomial $N_\delta(d)$.
begin
  Enumerate all templates Γ with $\delta(\Gamma) \leq \delta$;
  $N_\delta(d) \leftarrow 0$;
  forall *collections* $\tilde{\Gamma} = (\Gamma_1, \ldots, \Gamma_m)$ *with* $\sum_{i=1}^m \delta(\Gamma_i) = \delta$ **do**
    $i \leftarrow 1$;
    $Q_1 \leftarrow 1$;
    while $i \leq m$ **do**
      $a_i \leftarrow \max\Big(k_{min}(\Gamma_i), k_{min}(\Gamma_{i-1}) + l(\Gamma_{i-1}), \ldots, k_{min}(\Gamma_1) + l(\Gamma_1) + \cdots + l(\Gamma_{i-1})\Big)$;
    end
    while $i \leq m - 1$ **do**
      $Q_{i+1}(k_{i+1}) \leftarrow \sum_{k_i=a_i}^{k_{i+1}-l(\Gamma_i)} P(\Gamma_i, k_i) Q_i(k_i)$;
      $i \leftarrow i + 1$;
    end
    $Q^{\tilde{\Gamma}}(d) \leftarrow \sum_{k_m=a_m}^{d-l(\Gamma_m)+\varepsilon(\Gamma_m)} P(\Gamma_m, k_m) Q_m(k_m)$;
    $Q^{\tilde{\Gamma}}(d) \leftarrow \prod_{i=1}^m \mu(\Gamma_i) \cdot Q^{\tilde{\Gamma}(d)}$;
  end
  $N_\delta(d) \leftarrow N_\delta(d) + Q^{\tilde{\Gamma}}(d)$;
end

Algorithm 1: Algorithm to compute node polynomials.

Proof of Correctness of Algorithm 1. The algorithm is a direct implementation of Theorem 3.4. The m-fold discrete integral is evaluated symbolically, one sum at a time, using Faulhaber's formula (Lemma 3.5). Iterated symbolic integration affects the lower limits of the outer sums. The correct lower limit a_i of the ith sum is given by an iterated application of Lemma 3.6. $\square$

Remark 3.8. The running time of the algorithm can be improved vastly as follows: As the limits of summation in equation (3.1) only depend on $k_{min}(\Gamma_i)$, $l(\Gamma_i)$ and $\varepsilon(\Gamma_m)$, we can replace the template polynomials $P(\Gamma_i, k_i)$ by $\sum P(\Gamma_i, k_i)$, where the sum is over all templates Γ_i with prescribed $(k_{min}, l, \varepsilon)$. After this transformation the first sum in (3.1) is over all combinations of those tuples. This reduces the computation drastially as, for example, the 39769731 templates of cogenus 13 make up only 293 equivalence classes. Also, in equation (3.1) we can distribute the template multiplicities $\mu(\Gamma_i)$ and replace $P(\Gamma_i, k_i)$ by $\mu(\Gamma_i)P(\Gamma_i, k_i)$ and thereby eliminate $\prod \mu(\Gamma_i)$. Another speed-up is to compute all discrete integrals of monomials using Lemma 3.5 in advance.

Template enumeration seems to be the bottleneck of the algorithm. Their number grows exponentially with δ as can be seen from Figure 3. However, their generation can be parallelized easily (see below).

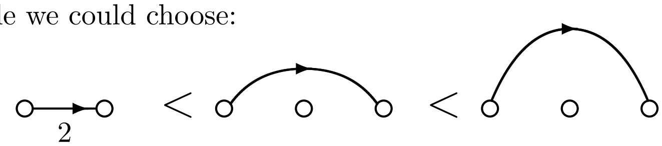
Algorithm 1 has been implemented in Maple [9]. Computing $N_{13}(d)$ on a machine with two quad-core Intel(R) Xeon(R) CPU L5420 @ 2.50GHz, 6144 KB cache, and 24 GB RAM took about 20 days.

Template Enumeration. To compute a list of all templates one can preceed as follows. First we need some terminology and notation. An edge $i \xrightarrow{e} j$ of a template is said to have *length* $j - i$. A template Γ is of *type* $\alpha = (\alpha_{ij})$, $i, j \in \mathbb{Z}_{>0}$, if Γ has α_{ij} edges of length i and weight j. Every type α satisfies, by definition of cogenus of a template,

$$\sum_{i,j\geq 1} \alpha_{ij}(i \cdot j - 1) = \delta(\Gamma). \tag{3.3}$$

Note that always $\alpha_{11} = 0$ as short edges are not allowed in templates. The number of types constituting a given cogenus δ via equation (3.3) is finite.

We can generate all templates of a ype α by the following "edge-sliding" branch and bound type algorithm which is illustrated, for $\alpha = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$, in Figure 3 . Let Γ_0 be the unique template of type α with all edges emerging from vertex 0. First we linearly order the set of edges of type α. In our example we could choose:



Start out with the template Γ_0 and generate graphs by iterating the following recursive procedure:

Let B be a graph that was obtained from a graph A by moving edge e_1. Choose an edge e_2 of B with $e_2 \geq e_1$ (if $B = \Gamma_0$ ignore this condition). Move e_2 to the next vertex if it creates a graph for which the natural order (from left to right) of the edges of the same type as e_2 can be extended to the fixed order that we chose at the beginning and call the resulting graph C. Do this for all permitted edges e_2 of B and apply this procedure to all resulting graphs C.

This recursion creates an infinite directed tree all of whose vertices correspond to different graphs. We eliminate a branch if either

(1) no edge of the root of the branch starts at vertex 1, or
(2) condition (3) in Definition 3.2 is impossible to satisfy for graphs further down the tree.

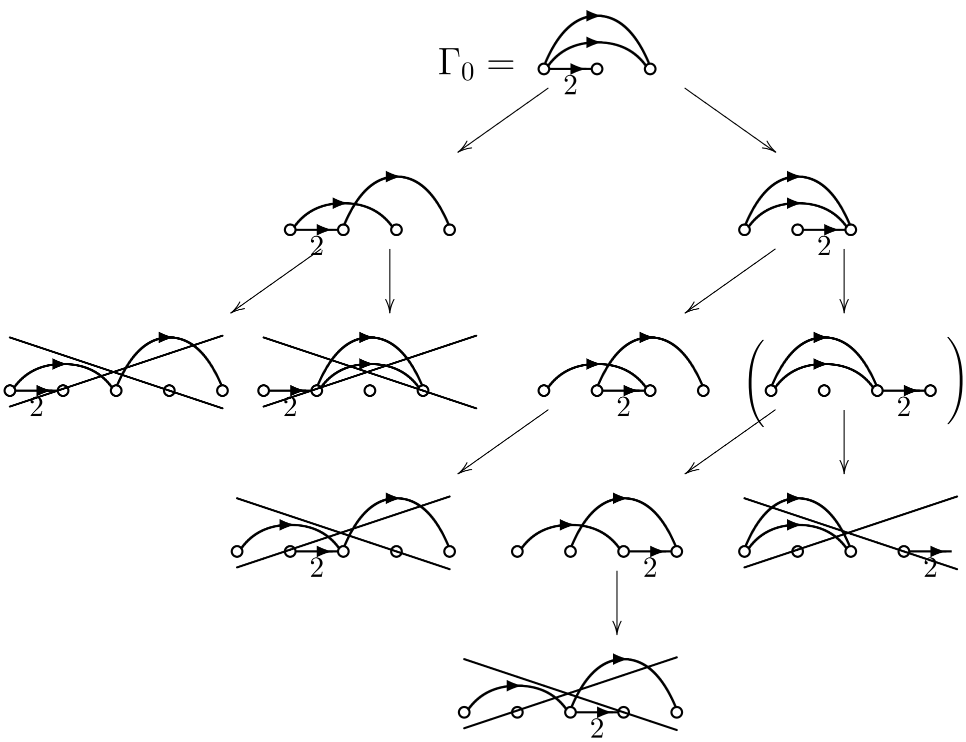


FIGURE 2. Branch and bound tree for $\alpha = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$.

A complete non-redundant list of all templates of type α is then given by all remaining graphs which satisfy condition (3) of Definition 3.2 as every template can be obtained in a unique way from Γ_0 by shifting edges in an order that is compatable with the order fixed earlier. Note that it can happen that a non-template graph precedes a template within a branch. See the graph in brackets in Figure 3 for an example. As enumerations for different types are independent they can be executed in parallel. The number of templates, for $\delta \leq 13$, is given in Figure 3.

There are many ways to generate all templates of a given cogenus δ. Ideally one would want a method that inductively generates templates from smaller templates as this seems to be most efficient. But as template combinatorics are tricky we did not yet succeed in producing a fast recursive algorithm that computes an exhaustive non-redundant list of templates of a given cogenus.

δ	# of templates	δ	# of templates
1	2	8	29913
2	7	9	125775
3	26	10	529755
4	102	11	2233572
5	414	12	9423100
6	1711	13	39769731
7	7135		

FIGURE 3. The number of templates grows exponentially with δ.

4. Threshold Values

In [6, Theorem 5.1] Fomin and Mikhalkin proved polynomiality of Severi degrees $N^{d,\delta}$ in d, for fixed δ, if d is sufficiently large. More precisely they showed that $N_\delta(d) = N^{d,\delta}$ for $d \geq 2\delta$. Here we show that their threshold value can be improved to $d \geq \delta$ (Theorem 1.2).

We need the following elementary observation about discrete antiderivatives of polynomials whose continuous counterpart is the well known fact that $\int_{a-1}^{a-1} f(x)dx = 0$.

Lemma 4.1. *For a polynomial $f(k)$ and $a \in \mathbb{Z}_{>0}$ let $F(n) = \sum_{k=a}^{n} f(k)$ be the polynomial uniquely determined by large enough values of n. ($F(n)$ is a polynomial by Lemma 3.5.) Then $F(a-1) = 0$. In particular, $\sum_{k=a}^{n} f(k)$ is a polynomial for $n \geq a-1$.*

Proof. Let $G(n)$ be the polynomial defined via $G(n) = \sum_{k=0}^{n} f(k)$ for large n. Then $F(n) = G(n) - \sum_{k=0}^{a-1} f(k)$ for all $n \in \mathbb{Z}_{\geq 0}$. In particular, $F(a-1) = G(a-1) - \sum_{k=0}^{a-1} f(k) = 0$. □

Proof of Theorem 1.2. By Lemma 3.6 and repeated application of Lemma 4.1 it suffices to show that $d \geq \delta$ simultaniously implies

$$\begin{aligned} d \geq\, & l(\Gamma_m) - \varepsilon(\Gamma_m) + k_{min}(\Gamma_m) - 1 \\ d \geq\, & l(\Gamma_m) - \varepsilon(\Gamma_m) + l(\Gamma_{m-1}) + k_{min}(\Gamma_{m-1}) - 2 \\ & \vdots \\ d \geq\, & l(\Gamma_m) - \varepsilon(\Gamma_m) + l(\Gamma_{m-1}) + \cdots + l(\Gamma_1) + k_{min}(\Gamma_1) - m \end{aligned} \tag{4.1}$$

for all collections of templates $(\Gamma_1, \ldots, \Gamma_m)$ with $\sum_{i=1}^{m} \delta(\Gamma_i) = \delta$.

By inspection we see that $l(\Gamma) - \varepsilon(\Gamma) \leq \delta(\Gamma)$ for all templates Γ, hence

$$l(\Gamma_m) - \varepsilon(\Gamma_m) - 1 \leq \delta(\Gamma_m) - 1$$

and

$$l(\Gamma_i) - 1 \leq \delta(\Gamma_i), \quad \text{for } 2 \leq i \leq m-1.$$

Also by inspection we observe

$$l(\Gamma_1) + k_{min}(\Gamma_1) - 1 \leq \delta(\Gamma_1) + 1$$

and the right hand side of the last inequality of (4.1) is $\leq \sum_{i=1}^{m} \delta(\Gamma_i) = \delta \leq d$. The proof of the other inequalities is very similar. □

5. Coefficients of Node Polynomials

The goal of this section is to present an algorithm for the computation of the coefficients of $N_\delta(d)$ for general δ. The algorithm can be used to prove Theorem 1.5 and thereby affirm and extend a conjecture of Di Francesco and Itzykso in [4] where they conjectured the seven terms of $N_\delta(d)$ of largest degree.

Our algorithm should be able to find formulas for arbitrarily many coefficients of $N_\delta(d)$. We will prove its correctness in this section. However, a proof of its termination is still missing. Still, termination for each instance can be checked (given enough CPU time). The algorithm is elementary in nature but there are a few subtleties.

First we fix some notation building on terminology of Section 3. By Remark 3.8 we can replace the polynomials $P(\Gamma, k)$ in equation (3.1) by the product $\mu(\Gamma)P(\Gamma, k)$, thereby removing the product of the template multiplicities. In this section we write $P(\Gamma, k)$ for $\mu(\Gamma)P(\Gamma, k)$. For integers $i \geq 0$ and $a \geq 0$ let $M_i(a)$ denote the matrix of the linear map

$$f(k) \mapsto \sum_{\Gamma:\delta(\Gamma)=i} \sum_{k=k_{min}(\Gamma)}^{n-l(\Gamma)} P(\Gamma, k) \cdot f(k), \tag{5.1}$$

where $f(k) = c_0 k^a + c_1 k^{a-1} + \cdots$, a polynomial of degree a, is mapped to $M_i(a)(f(k)) = d_0 n^{a+i+1} + d_1 n^{a+i} + \cdots$. (By Lemma 3.5 and the proof of Lemma 5.1 the image has degree $a + i + 1$). Hence $M_1(a)\mathbf{c} = \mathbf{d}$. Similarly, define $M_i^{\text{end}}(a)$ to be the matrix of the linear map

$$f(k) \mapsto \sum_{\Gamma:\delta(\Gamma)=i} \sum_{k=k_{min}(\Gamma)}^{n-l(\Gamma)+\varepsilon(\Gamma)} P(\Gamma, k) \cdot f(k). \tag{5.2}$$

Later we will consider square submatrices of $M_i(a)$ and $M_i^{\text{end}}(a)$ by restriction to the first N rows and columns which will be denoted $M_i(a)$ and $M_i^{\text{end}}(a)$ as well. Note that $M_i(a)$ and $M_i^{\text{end}}(a)$ are lower triangular. For a

large enough,

$$M_1(a) = \begin{bmatrix} \frac{6}{a+2} & 0 & 0 & 0 & 0 & \cdots \\ -\frac{5a+8}{a+1} & \frac{6}{a+1} & 0 & 0 & 0 & \cdots \\ \frac{5}{2}a+3 & -\frac{5a+3}{a} & \frac{6}{a} & 0 & 0 & \cdots \\ -\frac{1}{4}(4a+1)a & \frac{5}{2}a+\frac{1}{2} & -\frac{5a-2}{a-1} & \frac{6}{a-1} & 0 & \cdots \\ \frac{1}{40}(13a^2-20a+7)a & -a^2+\frac{7}{4}a-\frac{3}{4} & \frac{5}{2}a-2 & -\frac{5a-7}{a-2} & \frac{6}{a-2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In comparison

$$M_1(2) = \begin{bmatrix} \frac{3}{2} & 0 & 0 & 0 & 0 & \cdots \\ -6 & 2 & 0 & 0 & 0 & \cdots \\ 8 & -\frac{13}{2} & 3 & 0 & 0 & \cdots \\ -\frac{9}{2} & \frac{11}{2} & -8 & 0 & 0 & \cdots \\ 1 & -1 & 4 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

has finite support. The following observation is key to our algorithm.

Lemma 5.1. *The first $a+i$ rows of $M_i(a)$ and $M_i^{end}(a)$ are independent of the lower limits of summation in (5.1) and (5.2).*

Proof. It is an easy consequence of the proof of Lemma 5.7 of [6] that the polynomial $P(\Gamma, k)$ associated to a template Γ has degree $\leq \delta(\Gamma)$. On the other hand, if Γ is the template on vertices $0, 1, 2$ with i edges connecting 0 and 2 (so $\delta(\Gamma) = i$) equality is attained. As discrete integration of a polynomial increases the degree by 1 the polynomial on the right hand side of equation (5.1) has degree $1 + i + a$. □

The proof of this lemma implies that every denominator in row j of $M_i(a)$ is $a+i-j+2$ or 1 (after cancellation). As we will see later the basic idea of the algorithm is that templates with higher cogenera do not contribute to the higher degree terms of the node polynomial. With this in mind we define, for each finite collection $(\Gamma_1, \dots, \Gamma_m)$ of templates, its *type* $\tau = (\tau_2, \tau_3, \dots)$ where τ_i is the number of templates in $(\Gamma_1, \dots, \Gamma_m)$ with cogenus $\delta(\Gamma_s) = i$, for $i \geq 2$. Note that we do not record the number of templates with $\delta(\Gamma_s) = 1$.

Now we collect the contributions of all collections of templates with a given type. For each $\tau = (\tau_2, \tau_3, \dots) \in \mathbb{Z}_{\geq 0}^{\infty}$ with finite support and $\delta \geq \sum_{j \geq 2} \tau_j$ (so that there exist template collections $(\Gamma_1, \dots, \Gamma_m)$ of type τ with $\sum \delta(\Gamma_j) = \delta$) we define two (column) vectors $C_\tau(\delta)$ and $C_\tau^{\mathrm{end}}(\delta)$ as the coefficient vectors, listed in decreasing order, of the polynomials in n

(5.3)
$$\sum_{\Gamma_1,\dots,\Gamma_m} \sum_{k_m=k_{min}(\Gamma_m)}^{n-l(\Gamma_m)} P(\Gamma_m,k_m) \sum_{k_{m-1}=k_{min}(\Gamma_{m-1})}^{k_m-l(\Gamma_{m-1})} \cdots \sum_{k_1=k_{min}(\Gamma_1)}^{k_2-l(\Gamma_1)} P(\Gamma_1,k_1),$$

and

(5.4)
$$\sum_{\Gamma_1,\dots,\Gamma_m} \sum_{k_m=k_{min}(\Gamma_m)}^{n-l(\Gamma_m)+\varepsilon(\Gamma)} P(\Gamma_m,k_m) \sum_{k_{m-1}=k_{min}(\Gamma_{m-1})}^{k_m-l(\Gamma_{m-1})} \cdots \sum_{k_1=k_{min}(\Gamma_1)}^{k_2-l(\Gamma_1)} P(\Gamma_1,k_1),$$

where the respective first sums are over all ordered collections of templates of type τ.

It might look like $C_\tau(\delta)$ is a product of some matrices $M_i(a)$ applied to the polynomial 1. This is not the case! For example

$$C_{(0,0,\dots)}(2) = \begin{bmatrix} \frac{9}{2} \\ -34 \\ 88 \\ -\frac{179}{2} \\ 30 \\ 0 \\ \vdots \end{bmatrix} \neq \begin{bmatrix} \frac{9}{2} \\ -34 \\ 88 \\ -\frac{179}{2} \\ 27 \\ 0 \\ \vdots \end{bmatrix} = M_1(2)\cdot M_1(0)\cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

This is because, when iterated discrete integrals are evaluated symbolically, the lower limits of integration of the outer sums can change depending on the limits of the inner sums (compare with Lemma 3.6). This observation makes it necessary to compute initial values for recursions (described later) up to a large enough δ.

Before we can state the main recursion we need two more notations. For a type $\tau = (\tau_2, \tau_3, \dots)$ and $i \geq 2$ with $\tau_i > 0$ define a new type $\tau{\downarrow_i}$ via $(\tau{\downarrow_i})_i = \tau_i - 1$ and $(\tau{\downarrow_i})_j = \tau_j$ for $j \neq i$. Furthermore, let $\operatorname{def}(\tau) = \sum_{j\geq 2}(j-1)\tau_j$ be the *defect* of τ. The following lemma justifies this terminology.

Lemma 5.2. *The polynomials (5.3) and (5.4) are of degree $2\delta - \operatorname{def}(\tau)$.*

Proof. Let $(\Gamma_1,\dots,\Gamma_m)$ be a collection of templates with $\sum_{i=1}^m \delta(\Gamma_i) = \delta$ and type τ. Then, by the applying the argument in the proof of Lemma 5.1 to each Γ_i, the polynomials (5.3) and (5.4) have degree $\delta + m$. The result follows as

$$\begin{aligned}\delta - \operatorname{def}(\tau) &= \sum_{i=1}^{m} \delta(\Gamma_i) - \sum_{j\geq 2}(j-1)\tau_j \\ &= \sum_{i=1}^{m} \delta(\Gamma_i) - \sum_{j\geq 2}\left[\left(\sum_{i:\delta(\Gamma_i)=\tau_j} \delta(\Gamma_i)\right) - \tau_j\right] \\ &= \#\{i : \delta(\Gamma_i) = 1\} + \sum_{j\geq 2} \tau_j = m.\end{aligned}$$

□

The last lemma makes precise which collections of templates contribute to which coefficients of $N_\delta(d)$. Namely, the first N coefficients of $N_\delta(d)$ depend only on collections of templates with types τ such that $\operatorname{def}(\tau) < N$. The following recursion is the heart of the algorithm.

Proposition 5.3. *For every type τ and δ large enough, it holds that*

$$\begin{aligned} C_\tau(\delta) = &\sum_{i:\tau_i\neq 0} M_i\big(2\delta - i - 1 - \mathit{def}(\tau)\big)C_{\tau\downarrow i}(\delta - i) \\ &+ M_1\big(2\delta - 2 - \mathit{def}(\tau)\big)C_\tau(\delta - 1). \end{aligned} \tag{5.5}$$

More precisely, if we restrict all matrices M_i to be square of size $N - \mathit{def}(\tau)$ and all C_τ to be vectors of length $N - \mathit{def}(\tau)$, then the recursion holds for

$$\delta \geq \max\left(\left\lceil \frac{N+1}{2} \right\rceil, \sum_{j\geq 2} j\tau_j\right).$$

Proof. The coefficient vector $C_\tau(\delta)$ is defined by a sum that runs over all collections of templates $(\Gamma_1, \ldots, \Gamma_m)$ (see equation (5.3)). Partition the set of such collections by requiring that $\delta(\Gamma_m) = 1$, $\delta(\Gamma_m) = 2$ etc. This partitioning splits equation (5.3) exactly as in equation (5.5).

We can split off a matrix M_i if it does not affect the coefficient of the constant term (otherwise we cannot split if off as M_i does not allow variation of lower limits which is necessary by Lemma 3.6). If we can split then the polynomials (5.3) defining $C_{\tau\downarrow i}(\delta - i)$ and $C_\tau(\delta - 1)$ have degrees

$$2(\delta - i) - \operatorname{def}(\tau\downarrow i) = 2\delta - 2i - \operatorname{def}(\tau) + (i-1) = 2\delta - i - 1 - \operatorname{def}(\tau)$$

by Lemma 5.2 and, similarly, $2\delta - 2 - \operatorname{def}(\tau)$, respectively, which explains the arguments of the matrices M_1 and M_i. By Lemma 5.1, if $M_i(2\delta - i - 1 - \operatorname{def}(\tau))$ is of size $N - \operatorname{def}(\tau)$, then it does not depend on the lower limits if and only if $\delta \geq \frac{N+1}{2}$. In order for $C_\tau(\delta)$ to be defined (and the above identity to be meaningful) we need to impose $\delta \geq \sum_{j\geq 2} \tau_j$. □

Remark 5.4. Later, when we formulate the algorihm, we need to solve recursion (5.5) together with an initial condition in order to obtain an explicit formula for the first $N - \operatorname{def}(\tau)$ entries of $C_\tau(\delta)$. It suffices to take

$$\delta_0(\tau) \overset{def}{=} \max\left(\left\lceil \frac{N-1}{2} \right\rceil, \sum_{j\geq 2} j\tau_j\right)$$

as for any $\delta > \delta_0(\tau)$ the vector $C_\tau(\delta)$ of length $N - \operatorname{def}(\tau)$ can be written in terms of matrices M_i and vectors $C_{\tau'}(\delta')$ for various types τ' and $\delta' < \delta$.

We propose Algorithm 2 for the computation of the coefficients of the node polynomial $N_\delta(d)$. We explain the step which requires a solution of recursion (5.5) below.

Data: N, the number of coefficients of $N_\delta(d)$ we want to compute.
Result: C, the coefficient vector of the first N coefficients of $N_\delta(d)$.
begin
  Compute all templates Γ with $\delta(\Gamma) \leq N$;
  for *types* τ *with def*$(\tau) < N$ **do**
    Compute initial values $C_\tau(\delta_0(\tau))$ using (5.3), with $\delta_0(\tau)$ as in Remark 5.4;
    Solve recursion (5.5) for first $N - \operatorname{def}(\tau)$ coordinates of $C_\tau(\delta)$;

$$\text{Set} \quad C_\tau^{\text{end}}(\delta) \leftarrow \sum_{i:\tau_i\neq 0} M_i^{\text{end}}\big(2\delta - i - 1 - \operatorname{def}(\tau)\big)C_{\tau\downarrow i}(\delta - i) + M_1^{\text{end}}\big(2\delta - 2 - \operatorname{def}(\tau)\big)C_\tau(\delta - 1)$$

  end
  $C \leftarrow 0$;
  for τ *with def*$(\tau) < N$ **do**
    Shift the entries of $C_\tau^{\text{end}}(\delta)$ down by $\operatorname{def}(\tau)$;
    $C \leftarrow C +$ shifted $C_\tau^{\text{end}}(\delta)$;
  end
end

Algorithm 2: Algorithm to compute coefficients of the node polynomial.

Proof of Correctness of Algorithm 2. Proposition 5.3 guarantees that $C_\tau(\delta)$, the coefficient vector of contributions from template collections of type τ without the $\varepsilon(\Gamma)$-correction, is uniquely determined by recursion (5.3). By a similar argument as in the proof of Proposition 5.3 we see that $C_\tau^{\text{end}}(\delta)$ is given by the formula in Algorithm 2. By Lemma 5.2 all contributions of template collections of type τ to the node polynomial $N_\delta(d)$ are in degrees $\leq 2\delta - \operatorname{def}(\tau)$ of $N_\delta(d)$. Hence we need to shift $C_\tau^{\text{end}}(\delta)$ by $\operatorname{def}(\tau)$ before we sum these vectors. □

To solve recursion (5.5) for a type τ we make use of the following (conjectural) structure about $C_\tau(\delta)$ which has been varified for all types τ with $\operatorname{def}(\tau) \leq 7$.

Conjecture 5.5. *All entries of $C_\tau(\delta)$ are of the form $\frac{3^\delta}{\delta!}$ times a polynomial in δ.*

Now extend the natural partial order on the types τ given by $|\tau| = \sum_{j\geq 2} \tau_j$ to a linear order with smallest element $\tau = (0, 0, \dots)$. For example, for $N = 4$, we can take

$$(0,0,0) < (1,0,0) < (0,1,0) < (0,0,1) < (1,1,0) < (2,0,0) < (3,0,0).$$

Solve recursion (5.5) for each τ, in increasing order, using the lowertriangularity of the matrices M_i. For example, to compute the second entry $\frac{3^\delta}{\delta!}p(\delta)$ of $C_{1,1}(\delta)$ (assuming Conjecture 5.5), where $p(\delta)$ is a polynomial in δ, we need to solve

$$\begin{bmatrix} \vdots \\ \frac{3^\delta}{\delta!}p(\delta) \\ \vdots \end{bmatrix} = \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \vdots \\ \frac{3^{\delta-1}}{(\delta-1)!}p(\delta-1) \\ \vdots \end{bmatrix} + \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} * \\ * \\ \vdots \end{bmatrix} + \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} * \\ * \\ \vdots \end{bmatrix}$$

$$C_{1,1}(\delta) = M_1(2\delta-5)C_{1,1}(\delta-1) + M_2(2\delta-6)C_{0,1}(\delta-2) + M_3(2\delta-7)C_{1,0}(\delta-3)$$

The $*$-entries in the vectors $C_{0,1}$ and $C_{1,0}$ are known by a previous computation. Recall that all denominators of $M_i(a)$ in row j are $a+i-j+2$ or 1. To compute $p(\delta)$, or, more generally, the jth entry in $C_\tau(\delta)$, we first clear all denominators and then solve the polynomial difference equation with initial conditions

$$\begin{aligned} (2\delta - \operatorname{def}(\tau) - j + 1)3p(\delta) &= p(\delta-1) + q(\delta), \\ p(\delta_0(\tau)) &= C_\tau(\delta_0(\tau)), \end{aligned} \tag{5.6}$$

where $q(\delta)$ is a complicated polynomial depending on earlier calculations and $\delta_0(\tau)$ is as in Remark 5.4. One way to do this is to bound the degree of the polynomial $p(\delta)$ and solve the corresponding linear system.

Note that a difference equation of the form (5.6) need not have a solution in general. Conjecture 5.5 is equivalent to all recursions (5.6) appearing in Algorithm 2 to have a solution.

As in Section 3 Algorithm 2 can be improved vastly by combining the template polynomials $P(\Gamma, k)$ for templates Γ with fixed $\big(k_{min}(\Gamma), l(\Gamma), \varepsilon(\Gamma)\big)$.

Algorithm 2 has been implemented in Maple [9]. Once the templates are known the bottleneck of the algorithm is the initial value computation. With an improved implementation this should become faster than the template enumeration. Hence we expect Algorithm 2 to be able to compute the first 13 terms of $N_\delta(d)$ in reasonable time.

Appendix A. New Node Polynomials

An explicit list of $N_\delta(d)$ for $9 \le \delta \le 13$ is as below. These polynomials are given implicitly in Theorem 3.1.

$$\begin{aligned}
N_9(d) = {} & \frac{243}{4480}d^{18} - \frac{2187}{2240}d^{17} - \frac{729}{896}d^{16} + \frac{121743}{1120}d^{15} - \frac{99549}{280}d^{14} - \frac{824823}{160}d^{13} + \frac{8776593}{320}d^{12} + \frac{74122857}{560}d^{11} \\
& - \frac{2188424421}{2240}d^{10} - \frac{132610923}{70}d^{9} + \frac{11404136871}{560}d^{8} + \frac{2852923401}{224}d^{7} - \frac{3523392270287}{13440}d^{6} \\
& + \frac{4109675615}{448}d^{5} + \frac{261844582229}{128}d^{4} - \frac{2156232149611}{3360}d^{3} - \frac{29528525065861}{3360}d^{2} + \frac{438722045999}{168}d \\
& + 15580950065,
\end{aligned}$$

$$\begin{aligned}
N_{10}(d) = {} & \frac{729}{44800}d^{20} - \frac{729}{2240}d^{19} - \frac{729}{2240}d^{18} + \frac{408969}{8960}d^{17} - \frac{746253}{4480}d^{16} - \frac{1932579}{700}d^{15} + \frac{10649961}{640}d^{14} \\
& + \frac{205722099}{2240}d^{13} - \frac{4375229931}{5600}d^{12} - \frac{38815692777}{22400}d^{11} + \frac{30958937073}{1400}d^{10} + \frac{3413568339}{224}d^{9} \\
& - \frac{3624162885799}{8960}d^{8} + \frac{134470136581}{2800}d^{7} + \frac{27023302169081}{5600}d^{6} - \frac{22514488581251}{8960}d^{5} - \frac{811909836973903}{22400}d^{4} \\
& + \frac{253124357071961}{11200}d^{3} + \frac{867510616107447}{5600}d^{2} - \frac{2800250331071}{40}d - 283516631436,
\end{aligned}$$

$$\begin{aligned}
N_{11}(d) = {} & \frac{2187}{492800}d^{22} - \frac{2187}{22400}d^{21} - \frac{729}{6400}d^{20} + \frac{150903}{8960}d^{19} - \frac{303993}{4480}d^{18} - \frac{56670273}{44800}d^{17} + \frac{47717667}{5600}d^{16} \\
& + \frac{295979589}{5600}d^{15} - \frac{11410430877}{22400}d^{14} - \frac{4051907631}{3200}d^{13} + \frac{52491198663}{2800}d^{12} + \frac{3418059518271}{246400}d^{11} \\
& - \frac{20587006282467}{44800}d^{10} + \frac{2236636275459}{22400}d^{9} + \frac{49175916627959}{6400}d^{8} - \frac{1464110674563}{256}d^{7} \\
& - \frac{1946239824069277}{22400}d^{6} + \frac{3767687640687823}{44800}d^{5} + \frac{14264414890838423}{22400}d^{4} - \frac{940418544772283}{1600}d^{3} \\
& - \frac{168280746183263029}{61600}d^{2} + \frac{5073050867636909}{3080}d + 5187507215325,
\end{aligned}$$

$$\begin{aligned}
N_{12}(d) = {} & \frac{2187}{1971200}d^{24} - \frac{6561}{246400}d^{23} - \frac{2187}{61600}d^{22} + \frac{496449}{89600}d^{21} - \frac{136809}{5600}d^{20} - \frac{1618623}{3200}d^{19} + \frac{674946837}{179200}d^{18} \\
& + \frac{2321658693}{89600}d^{17} - \frac{893195181}{3200}d^{16} - \frac{34334301951}{44800}d^{15} + \frac{289702847403}{22400}d^{14} + \frac{1245724147341}{123200}d^{13} \\
& - \frac{803786361621603}{1971200}d^{12} + \frac{65497548165237}{492800}d^{11} + \frac{16192295343681}{1792}d^{10} - \frac{792669234543351}{89600}d^{9} \\
& - \frac{9506773589164709}{67200}d^{8} + \frac{6296062244021929}{33600}d^{7} + \frac{11029935159768347}{7168}d^{6} - \frac{582428855393100577}{268800}d^{5} \\
& - \frac{5477484616918678589}{492800}d^{4} + \frac{10067756533588172119}{739200}d^{3} + \frac{4454424013895459501}{92400}d^{2} \\
& - \frac{111952943233924509}{3080}d - 95376705265437,
\end{aligned}$$

$$\begin{aligned}
N_{13}(d) = {} & \frac{6561}{25625600}d^{26} - \frac{6561}{985600}d^{25} - \frac{19683}{1971200}d^{24} + \frac{1620567}{985600}d^{23} - \frac{88209}{11200}d^{22} - \frac{3212703}{17920}d^{21} + \frac{262066023}{179200}d^{20} \\
& + \frac{494726373}{44800}d^{19} - \frac{673360047}{5120}d^{18} - \frac{35350103511}{89600}d^{17} + \frac{20952637821}{2800}d^{16} + \frac{3013479294723}{492800}d^{15} \\
& - \frac{580214902388013}{1971200}d^{14} + \frac{1666286215401123}{12812800}d^{13} + \frac{16384163286402207}{1971200}d^{12} - \frac{909876952033137}{89600}d^{11} \\
& - \frac{7649416285706767}{44800}d^{10} + \frac{25855007471662161}{89600}d^{9} + \frac{65085797443981191}{25600}d^{8} - \frac{108443195356282427}{22400}d^{7} \\
& - \frac{52991400162927629917}{1971200}d^{6} + \frac{19763246047110315 17}{39424}d^{5} + \frac{135807530802431052 19}{70400}d^{4} \\
& - \frac{7327470596743106328 1}{246400}d^{3} - \frac{68173290776099374391}{80080}d^{2} + \frac{2813974748454890667}{3640}d + 1761130218801033.
\end{aligned}$$

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