Superattractive Fixed Points in $\mathbb{C}^n$

$$\begin{tikzcd} F: \mathbb{C}^n \arrow[r] \arrow[d, "\pi"'] & \mathbb{C}^n \arrow[d, "\pi"] \\ f: \mathbb{P}^{n-1} \arrow[r] & \mathbb{P}^{n-1} \end{tikzcd}$$

$z \longmapsto F(z)$ homogeneous polynomial, degree d, $F^{-1}(0) = 0$.

$z \longmapsto f(z)$ holomorphic map.

$$\Omega := \{ F^{\circ n}(z) \to 0 \}.$$

Goal: 1) Show ~~that~~ the smooth locus of $\partial\Omega := \partial\Omega_{sm}$ is Levi-flat ie foliated by complex curves.

2) Describe foliation
- holonomy
- topologically

Thm: $h_F(z) := \lim_{n\to\infty} \frac{1}{d^n} \log|F^{\circ n}(z)|$

- $h_F : \mathbb{C}^n \to [-\infty, \infty)$ is psh.
- $\Omega = \{h_F < 0\}$, $\partial\Omega = \{h_F = 0\}$. ($\Rightarrow \partial\Omega$ circled, i.e. $S^1 \partial\Omega = S^1$)
- $F_f :=$ domain of normality for f on $\mathbb{P}^{n-1}$, then h_F is pluriharmonic on $\pi^{-1}(F_f) \setminus \{0\}$

Pf: Conjugate F w/ multiplication by $t \in \mathbb{R}$

$\Rightarrow$ may assume $\frac{1}{C}|z|^d \leq |F(z)| \leq |z|^d$

$$\frac{1}{C}|F^{\circ(n-1)}(z)|^d \leq |F^{\circ n}(z)| \leq |F^{\circ(n-1)}(z)|^d$$

$$\Rightarrow \frac{1}{d^{n-1}} \log|F^{\circ n-1}(z)| - \frac{\log(C)}{d^n} \leq \frac{1}{d^n}\log|F^{\circ n}(z)| \leq \frac{1}{d^{n-1}} \log|F^{\circ(n-1)}(z)|.$$

- Decr sequence of psh functions is psh.
- Convergence is <u>uniform</u>

  (when δ^{on} normal on $V \ni z$, $\underset{\text{can}}{f^{o\gamma} \to g}$ instead use a norm $\|\cdot\|$ s.t. $\log\|\cdot\|$ pluriharmonic on $g(W)$ shrinking V if necessary).

  $\Rightarrow h_f$ pluriharmonic on $\pi^{-1}(F_f) \setminus \{0\}$.

<u>So:</u> $\partial\Omega = \{h_f = 0\}$ smooth above F_f.

h_f pluriharmonic $\Rightarrow \partial\Omega_{sm}$ Levi flat

i.e. $\sum \frac{\partial^2 h_f}{\partial z_i \partial \bar{z}_j} \equiv 0$.

$\Rightarrow T(\partial\Omega) \cap iT(\partial\Omega)$ integrable cx system

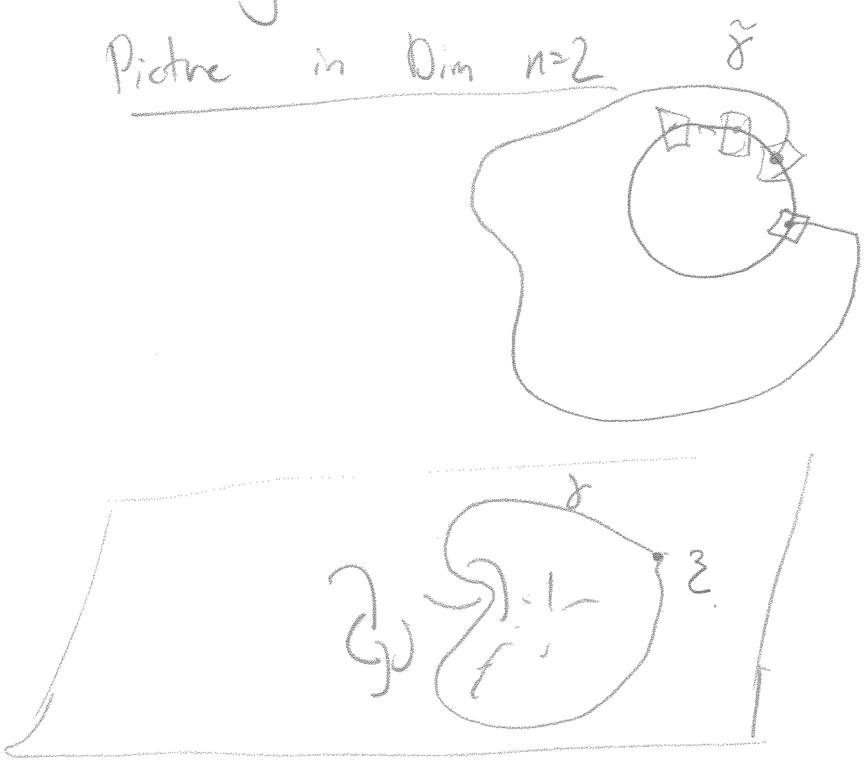
$\Rightarrow \partial\Omega_{sm}$ foliated by cx manifolds.

<u>Picture in Dim $n=2$</u>

(z_1, z_2)

$\downarrow$

$\zeta = \frac{z_1}{z_2}$



Let $r(z) :=$ radius of $\partial\Omega \cap \pi^{-1}(z)$.

Claim: $\log\left(\frac{r(z)}{\sqrt{1+|z|^2}}\right)$ harmonic on F_f

Pf: $h_F\left(\frac{r(z)}{\sqrt{1+|z|^2}}(z,1)\right) \equiv 0$

$$\|$$

$h_F(z,1) + \log\left(\frac{r(z)}{\sqrt{1+|z|^2}}\right) \equiv 0$ Take $\partial\bar\partial$.

So $\log\left(\frac{r(z)}{\sqrt{1+|z|^2}}\right) = u$ harmonic

Pick local harmonic conjugate v

$\Rightarrow$ Foliation given by

$$z \mapsto \left(z e^{u+iv}, e^{u+iv}\right)$$

If γ is closed curve in F_f,

monodromy given by $e^{i\int_\gamma * du}$

i.e. If $\tilde\gamma(0) = \left(z e^{u+iv}, e^{u+iv}\right)$

$\tilde\gamma(1) = \left(z e^{u+i(v+\int_\gamma * du)}, e^{u+i(v+\int_\gamma * du)}\right)$

Assume $f(z)$ monic poly. $\Rightarrow$ $F(z_1, z_2) = (f(z_1), z_2^d)$

Def: $\varphi :=$ Böttcher coord. $\qquad G_f$ - Green's function.

$\varphi(z) := \lim\limits_{z \to \infty} f^{\circ n}(z)^{\frac{1}{d^n}} \qquad G_f(z) = \lim\limits_{n\to\infty} \frac{1}{d^n} \log |f^{\circ n}(z)|$

- Green's function of f.

Satisfies. $\qquad$ ($\overset{\text{Thm}}{=}$ equilibrium measure of J_f).

$\varphi(f(z)) = \varphi(z)^d$

(in nbhd of ∞)

Prop: $h_F(z_1, z_2) = G_f\left(\frac{z_1}{z_2}\right) + \log|z_2|$

Pf. $F^{\circ n}(z_1, z_2) = z_2^{d^n}\left(f^{\circ n}\left(\frac{z_1}{z_2}\right), 1\right)$

$\frac{1}{d^n} \log|F^{\circ n}(z_1, z_2)| = \frac{1}{d^n}\log\left|f^{\circ n}\left(\frac{z_1}{z_2}\right), 1\right| + \log|z_2|$

If $\frac{z_1}{z_2} \in K_f \qquad \to 0 \Rightarrow$ formula works.

If $\frac{z_1}{z_2} \notin K_f$, $\quad \frac{1}{d^n}\log\left|f^{\circ n}\left(\frac{z_1}{z_2}\right), 1\right| \to G_f\left(\frac{z_1}{z_2}\right)$

Prop: For $|a| = 1$, foliation of $\partial\Omega_{sm}$ parametrized by

$\gamma_a(z) = \left(\frac{a\varphi^{-1}(z)}{z}, \frac{a}{z}\right)$

Pf. Just n.t.s. $h_F(\gamma_a(z)) = 0$

$= G_f(\varphi^{-1}(z)) + \log\left|\frac{a}{z}\right|$

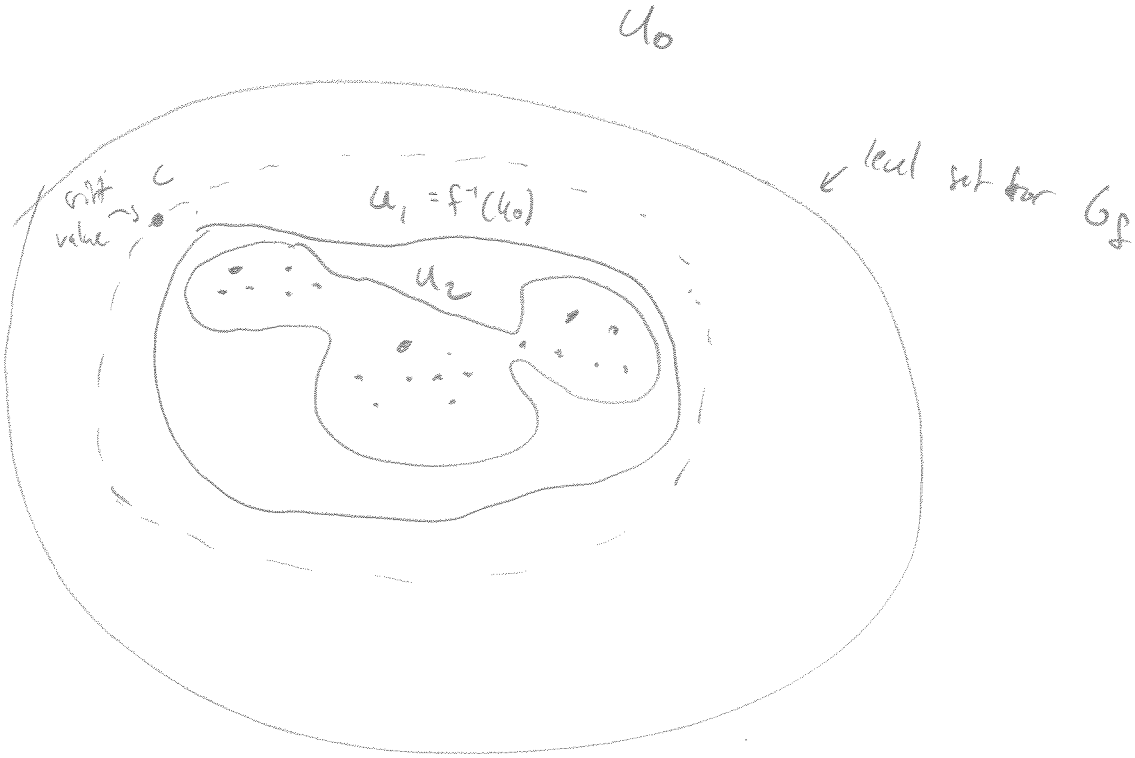
$\log|z| - \log|z| + \log|a| = 0.$

Rmk: K_f connected, then $\partial\Omega_{sm} \cap C((\mathbb{P}^1 \setminus K_f)$ are just analytic disks (Böttcher coord. exists on all of $\mathbb{P}^1 \setminus K_f$, analytic disk)

<u>Prop:</u> If $f(z) = z^2 + c$, K_f not connected then $C(\mathbb{P}^1 \setminus K_f) \cap \partial\Omega$ is foliated by Riemann surfaces of infinite genus, each of which is <u>dense</u>

· crit value/pt. @ ∞

Pf:



Foliation above U_0 given by $X_{\bar{a}} := \gamma_{\bar{a}}(\overline{U_0})$.

Set $X^i_a = F^{-1}(X_a)$ - foliation above U_i.

<u>Lemma:</u>

a) X^i_a is connected, $X^i_a \supseteq \bigcup_{a'^{2^i} = a^{2^i}} X_{a'}$

b) X^i_a contains 2^{i+1} crit value of F, above each are 2 ordinary double points

c) Each X^i_a has 2^i boundary components.

Sketch: $F(\gamma_a(z)) = \gamma_{a^2}(z^2)$

$\Rightarrow F: X_a^i \to X_{a^2}^{i-1}$ ramified covering, degree 4, double # boundary components @ each step.

Rieman-Hurwitz

$$\Rightarrow \chi(X_a^i) = 4\cdot\chi(X_{a^2}^{i-1}) - 2^{i+1}$$

$$\chi(X_a^0) = 1$$
(disk)

$$\Rightarrow \chi(X_a^i) = -2^{i+1}(2^{i-1}-1)$$

$$\Rightarrow g(X_a^i) = 1 - \tfrac{1}{2}(\chi(X_a^i) + \#\partial\text{components}(X_a^i))$$
$$= 1 + 2^{i-1}(2^i - 3) \to +\infty$$