Pondering the Projective Plane

Goals: Define a "geometry" and the Projective Plane

- Give some ways to think about it and why it's worth studying.

**Def:** Plane Geometry

- points (underlying set)
- lines (subset collection of subsets)
- axioms (rules)

Ex: Euclidean Plane

Euclid - "5" axioms

Hilbert - "20"

5th axiom - Parallel Postulate

Given a line L, p \& L, 3rd line through p not intersecting L

"mysterious" "less constructive" "not the 5th"

Q: Can you derive the 5th axiom from the other 4?

No - there exist geometries satisfying the first 4, not the 5th.
Spherical Geometry
- points - points on a sphere
- lines - great circles on a sphere.

Parallel Postulate States
Every pair of lines intersects at 2 points
\implies no parallel lines

Hyperbolic Geometry (talk to Allenby / Margulis / anyone)
- points are \( \mathbb{R}^2 \setminus \{0\} \)
- lines are lines + circles intersecting unit circle at 90°

Parallel Postulate States
Given \( p \neq L \), 3 infinitely many "parallel" lines through \( p ")
- draw

Projective Geometry (talk to San Payne / anyone)
"nicest" possible geometry "containing" Euclidean Geometry
Want: Every pair of lines intersects in exactly one point.
(Parallel postulate false)

Failure in Euclidean Geometry "at infinity"

\[ \ell_1, \ell_2 \]

\( \ell_1 \parallel \ell_2 \), \( \ell_2 \parallel \ell_3 \)

\( \ell_2 \parallel \ell_3 \) "point at infinity" for lines parallel to \( \ell_1 \)

Fact: \( \text{Area}(\Delta) = \alpha + \beta + \gamma - \pi \)
\( \implies \text{sum of angles} \geq \pi \)

Fact: \( \text{Area}(\Delta) = \pi - (\alpha + \beta + \gamma) \)
\( \implies \text{sum of angles} \leq \pi \)
§ Projective Plane

- Points - Euclidean Plane $U \in \mathbb{E}^2$ | $L$ a line $\mathbb{P}^1$
- Lines - $L \in \mathbb{E}^2 \setminus \mathbb{P}^1$ - $L$ a Euclidean line.
  - "line at infinity" - $U \in \mathbb{E}^2 \setminus \mathbb{P}^1$ - want there to exist a line through every 2 pts.

Diagram:

- Line at infinity "glued" in a weird way.

Check:
- 3! line through every pair of points
- Every 2 lines intersect at exactly one point.

Pappus Thm: (Euclidean)
Let $A, B, C$ collinear; $a, b, c$ collinear. Then

$AC' \cap CA', \ AB' \cap BA', \ BC' \cap CB'$

are collinear unless one pair of lines is parallel
in which case bad stuff.
"Generic" "Good Case"

"Bad Case"
Pappus' Thm (Projective)

Let $A, B, C$ be collinear, $A', B', C'$ collinear.
Then $\overline{AB} \cap \overline{B'A'}$, $\overline{AC} \cap \overline{C'A'}$, $\overline{BC} \cap \overline{C'B'}$ are collinear.

§ Projective Coordinates

"Model" projective plane as

- Vector space $- \mathbb{R}^3$
  - "points" - 1 dim subspaces - line through origin
  - "lines" - 2 dim subspaces - plane through origin

Check axioms

- 3! "line" through 2 pts

- 2 lines intersect in exactly 1 pt

- 3! 2 dim subspaces containing 2 1-dim subspaces

- 2 2dim subspaces intersect at a unique 1-dim subspace

$p$ at infinity.
Ponies

This model generalizes to any field.

- Gives you a group of symmetries.
- $3 	imes 3$ matrices act on $\mathbb{R}^3$ - all points are "the same".

**Dualities**: turn "points" into "lines" and preserve incidence structure.

Claim: Orthogonal complement works.

- $p$ 1-dim subspace $\rightarrow p^\perp$ 2-dim subspace
- $e$ 2-dim subspace $\rightarrow e^\perp$ 2-dim subspace.

If I have a theorem, I get a dual theorem for free!

**Pappus (Dual) Theorem**: If lines $A, B, C$ intersect at a point,

- lines $A', B', C'$ intersect at a point

then

\[
\frac{A_0 B}{B_0 A'} = \frac{A_0 C}{C_0 A'} = \frac{B_0 A'}{B_0 C'}
\]

intersect at 1 point.
Pappus' Dual Theorem

5" Real" Planar Graphs

Def: Planar graph is a graph you can draw in the Euclidean plane with non-intersecting edges.

Ex: $K_4$

Fact: $K_5$ is not planar.

Why? Euler's Formula

$V - E + F = 2$
Q: Is $k_S$ "planar" in projective plane?

Idea: Use extra lines at infinity to make it planar.

Q1: Is $k_6$ "real" planar? Yes!

Q2: "" $k_7$ ""?

Q3: Why does Euler's formula not fail?

If time:

- $L_1$ always above $L_2$

$\quad L_1$

$\quad L_2$

- should cross at intersection point.

Q: What's going on here?