Complex Tori + Lattices Maps:

Goal: Define lattice maps, show why they're interesting.

Def: Lattice Map
\[ T \rightarrow T \]
\[ T \rightarrow T \]
\[ \mathbb{C} \rightarrow \mathbb{C} \]
\[ \tau \rightarrow \tau \]
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Part 2: Contact Map \( T \to \hat{\mathbb{C}} \)

\[ T \to T_{z \mp z^3} \]

Locally injective except when \( z = -z \) (A)

\[ 2z = 0 \] (A)

\[ z \in \mathbb{R} \nabla \]

4 \* "bit phi" where map \( \phi \) "looks like" \( z \to z^2 \).

Compute genus of \( T_{z \mp z^3} \) using Poincaré-Hinze.

\[ \Theta: T \to T_{z \mp z^3} = S \]

\[ \sum_{x \in T} (\text{mult}_x T - 1) = \chi(S) \cdot \deg(\theta) - \chi(T) \]

\[ 4 = 2 \cdot \chi(S) - 0 \]

\[ \Rightarrow \chi(S) = 2 \]

\[ 2 - 2g \Rightarrow g = 0 \quad \text{i.e. S is a sphere} \approx \mathbb{C} \]

So have a map \( \Theta: T \to \hat{\mathbb{C}} \) \( \Theta(z) = \Theta(-z) \).

\( \Theta \) has name: \( \varphi \) - function (up to normalization).

Canonical map (up to Möbius Trans).
Part 3: How to get \( f \).

Consider map: \( L: \mathbb{C}/i \rightarrow \mathbb{C}/i \), \( z \rightarrow 2iz \).

\[ L(-z) = -L(z) \]

Thus get well defined map in \( T/2 \mathbb{Z} \), \( T \subset \mathbb{C} \).

\[ \mathbb{C}/i \xrightarrow{L} \mathbb{C}/i \]

\[ \Theta \downarrow \quad \downarrow \quad \Theta \]

\[ f: \mathbb{C} \rightarrow \mathbb{C} \]

\[ f \text{ is "interestingly"} \]

1. Periodic pts are dense for \( L \)

2a. \( L^n(z) = z \Rightarrow (L^n)'(z) = 2^n \)

\[ \Rightarrow \text{ Periodic pts dense for } f \]

2b. \( f^n(z)^n \Rightarrow (f^n)'(z) = 2^n \).

1a. Look at \( \frac{2i}{p} + \frac{2i}{p} \mathbb{Z} \) \( p \) odd prime. \( 0 \leq i < p \).

\[ \frac{2i}{p} \equiv \frac{2i}{p} \mod p \]

Since \( 2 \) is a unit mod \( p \), \( 2^i \equiv 1 \) for some \( i \)

\[ \Rightarrow \text{ periodic.} \]

2b. Exer.
(b) \( L^n(z) = z \) & periodic pt.

\[ \implies f^n(\theta(z)) = \theta(L^n(z)) = \theta(z) \]

\[ \implies \theta(z) \text{ periodic pt.} \]

To compute derivative:

\begin{align*}
\theta' & \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 & \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\theta & \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\theta & \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\theta & \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\end{align*}

\( \theta \) has local move away from \( z \).

are there is local coords to compute derivatives.

Conclusion: \( f \) dense set of periodic pts \( \implies \)

unbounded derivatives.

As Julia set = \( \mathbb{C} \) all of \( \mathbb{C} \).

(chaotic set).

What explicit formula for \( f \) in this case:

Fact: \( L \) is 4-to-1 (degree 4)

\[ \implies f \text{ is 4-to-1 (degree 4) \& exactly 4 poles (w/ mult.)} \]

Composing \( \theta \) mobius transformation to get \( \theta \circ T = \mathbb{C} \) s.t.

\[ \theta(0) = \infty \quad \theta(\frac{1}{2}) = 0 \]

\[ \theta(z) = 1 \quad \theta\left(\frac{1+\xi}{2}\right) = a(z) \text{ (mobius form)} \]

\[ \theta(\infty) = 1 \]

\[ \theta(0) = \infty \]

\[ \theta(1) = 0 \]

\[ \theta\left(\frac{1+\xi}{2}\right) = a(z) \text{ (mobius form)} \]
Claim: \( \text{poles (f)} = 0, 1, \infty, a \).

\[
T \to T \\
T \to T
\]

\[
f(0) = f(f(0)) = f(0) = \infty
\]

\[
f(1) = f(f(\frac{1}{2})) = f(\frac{1}{2}) = \frac{1}{2}
\]

\[
e^T \cdot e
\]

Claim: \( \text{zeros (f)} = \pm 3a \).

Consider \( \tilde{h} : T \to T \) \( \tilde{h}(z) = z + \frac{1}{2} \) \( \text{(2)} \)

\[
h : \mathbb{C} \to \mathbb{C} \\
h(z) = \frac{a}{z}
\]

Claim: \( h(g(z)) = g(h^{-1}(z)) \) \( \text{Ex: Similar theory.} \)

\[
\omega + i \\
f(\omega) = 0 \Rightarrow h(\omega) = \omega.
\]

\[
\omega = g(z) \Rightarrow f(g(z)) = g(2z) \Rightarrow 2z = \frac{1}{2} \text{ (4)}
\]

\[
h(\omega) = h(g(\omega)) = \frac{g(\omega)}{g(\omega) - 1} = g\left( \frac{1}{4} \right) = g\left( \frac{1}{4} \right) = \frac{1}{4}
\]

Claim: \( \pm 3a \) \( \text{are zeros of mult. 2.} \)

Ex: Similar theory \( w' \to z + \frac{z}{2} \)

\[
f(z) = C \cdot \frac{(z-a)^2}{\omega(z-1)(z-a)} \Rightarrow f(\omega) = \frac{(\omega^2 - a)^2}{4\omega(\omega-1)(\omega-a)}
\]

\[
f'(\omega) = 4
\]

\[
C = 4
\]

\[
T \to T \\
T \to T
\]

\[
\tilde{h} \to e
\]

\[
e^T \cdot e
\]

\[
T \to T \\
T \to T
\]

\[
\mathbb{C} \to \mathbb{C}
\]

\[
e^T \cdot e
\]

\[
\mathbb{C} \to \mathbb{C}
\]

\[
(2a)^2 = 4 \times 2
\]
Main: If we vary $z$ ($A = \langle 1, z \rangle$)

$\Rightarrow$ vary $a$, get

1 parameter family of Lattes maps.

All "flexible" Lattes maps look like.

$$ T \mapsto T \quad \text{deform Torus.} $$

$$ \hat{\mathcal{C}} \mapsto \mathcal{C} = T/\mathbb{Z} \times \mathbb{R} $$

"rigid" Lattes maps:

$$ A = \langle 1, i \rangle \quad \text{special since } i \cdot A = A. $$

$$ T \mapsto T $$

Cannot "deform" torus

since need to

keep condition $i \cdot A$.

$$ \hat{\mathcal{C}} \mapsto \hat{\mathcal{C}} = T/\mathbb{Z} \times \mathbb{R} \times \mathbb{R} $$

$$ A = \langle 1, 2, \xi \rangle \quad T \mapsto \hat{\mathcal{C}} $$

$$ T/\mathbb{Z} \times \mathbb{Z}_{2} \times \mathbb{Z}_{2} $$

or

$$ T/\mathbb{Z} \times \mathbb{Z}_{3} \times \mathbb{Z}_{3} $$
I.e. Modding out $\mathbb{Z}^n$ by $n^{th}$ roots of unity.

Claim: $n = 2, 3, 4, 6$

Let $\mathbb{Z}^n$ be a field of vectors of non-zero vector is 1. Need to have $2n+1 = 1$. For all $n$

Assume $n$ is even (since $A$ closed under $\mathbb{Z}^n$).

Claim: $\text{# smallest vectors is } \leq 6$.

May assume $n$ even since $A$ closed under $\mathbb{Z}^n$.

If $\text{#} = 8$, contradiction.

Circumference of circle is $2\pi$, so if had $\geq 8$ smallest vectors, 2 of them would be distance \( \pi \) apart on circle, \( \Rightarrow \langle v, v \rangle \text{ smaller, contradiction} \)

$\Rightarrow \text{# smallest set of vectors } = 6$. 