Complex Tori + Lattès Maps:

Goal: Define Lattès maps, show why they're interesting.

Def: Lattès Map

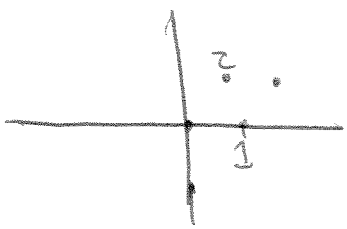
$$\begin{tikzcd} T \arrow[r, "L"] \arrow[d, "\theta"'] & T \arrow[d, "\theta"] \\ \hat{\mathbb{C}} \arrow[r, "f"] & \hat{\mathbb{C}} \end{tikzcd}$$

- T - torus
- everything is holomorphic.

f is the Lattès map.

Part 1.

Def Clx Torus $\mathbb{C}/\Lambda$. $\Lambda = \langle 1, \tau \rangle$.



Prop: $L : \mathbb{C}/\Lambda \longrightarrow \mathbb{C}/\Lambda \Rightarrow L(\omega) = \alpha\omega + \beta \; (\Lambda)$.

Pf:

$$\begin{tikzcd} \tilde{L} : \mathbb{C} \arrow[r] \arrow[d] & \mathbb{C} \arrow[d] \\ L : \mathbb{C}/\Lambda \arrow[r] & \mathbb{C}/\Lambda \end{tikzcd}$$

$g(z) = \tilde{L}(z) - \tilde{L}(z+\gamma)$ $\quad \gamma \in \Lambda$.

$\mathrm{Image}(g) \in \Lambda$.

$\mathbb{C}$ con. $\Rightarrow g(\mathbb{C}) = \gamma$

$\Rightarrow$ i.e. g const $\Rightarrow g' = 0$

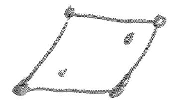
$\Rightarrow \tilde{L}'(z) = \tilde{L}'(z+\gamma) \quad \forall \gamma \in \Lambda$.

$\Rightarrow \tilde{L}'$ doubly periodic

Louisville $\Rightarrow \tilde{L}' = \alpha \Rightarrow \tilde{L}(z) = \alpha z + \beta$ ...

Part 2: Construct Map $T \xrightarrow{\theta} \hat{\mathbb{C}}$

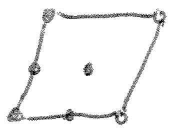
$$T \xrightarrow{\theta} T/\{\pm z\}$$



locally injective except where $z = -z \ (\Lambda)$

$$\Rightarrow 2z \equiv 0 \ (\Lambda)$$

$$z \in \tfrac{1}{2}\Lambda.$$



4 "crit pts" where map "looks like" $z \mapsto z^2$.

Compute genus of $T/\{\pm z\}$ using Riemann-Hurwitz.

$$\theta : T \to T/\{\pm z\} = S$$

R-H Formula

$$\sum_{x \in T} (\text{mult}_x T - 1) = \chi(S) \cdot \deg(\theta) - \chi(T).$$

$$4 = 2 \cdot \chi(S) - 0$$

$$\Rightarrow \chi(S) = 2$$

$$2-2g \Rightarrow g = 0. \quad \text{i.e. } S \text{ is a } \underline{\text{sphere}} \cong \hat{\mathbb{C}}.$$

So have holomorphic map $\theta : T \longrightarrow \hat{\mathbb{C}}$ $\quad \theta(z) = \theta(-z)$.

θ has name: $\wp$-function (up to normalization).

: canonical map (up to Möbius trans).

Part 3: How to get f.

Consider map: $L : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$ $\quad z \mapsto 2z$.

$$L(-z) = -L(z)$$

$\Rightarrow$ get well defined map in $T/\{\pm z\} = \hat{\mathbb{C}}$.

$$\begin{tikzcd} \mathbb{C}/\Lambda \arrow[r, "L"] \arrow[d, "\theta"'] & \mathbb{C}/\Lambda \arrow[d, "\theta"] \\ f : \hat{\mathbb{C}} \arrow[r] & \hat{\mathbb{C}} \end{tikzcd} \qquad \theta \circ L = f \circ \theta$$

f is "interesting":

1)a Periodic pts are dense for L.

2)a $L^{\circ n}(z) = z \Rightarrow (L^{\circ n})'(z) = 2^n$

$\Rightarrow$ 1b) Periodic pts dense for f

2b) $f^{\circ n}(z) = z \Rightarrow (f^{\circ n})'(z) = 2^n$.

1a). Look @ $\frac{i}{p} + \frac{j}{p}\tau$ $\quad$ p odd prime. $0 \le i, j < p$.

 $\longmapsto \frac{2i}{p} + \frac{2j}{p}\tau$ - reduce the numerators mod p.

p^2 pts.

Since 2 is a unit mod p, $2^n \equiv 1$ for some n

$\Rightarrow$ periodic.

2b. Exer:

1(b). $L^n(z) = z$. z periodic pt.

$\Rightarrow f^n(\theta(z)) = \theta(L^n(z)) = \theta(z)$

$\Rightarrow \theta(z)$ periodic pt.

To compute derivative.

$$\begin{tikzcd}
z_1 \arrow[r, "L"] \arrow[d, "\theta"'] & z_2 \arrow[r, "L"] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r, "L"] \arrow[d] & z_1 \arrow[d, "\theta"] \\
w_1 \arrow[r, "f"] & \cdot \arrow[r] & \cdot \arrow[r] & \cdot \arrow[r] & w_1
\end{tikzcd}$$

θ has local inverse away from $\frac{1}{2}\Lambda$

use these as local coords to compute derivatives.

Conclusion: f: dense set of periodic pts w/ unbounded derivatives.

$\Rightarrow$ Julia set = ~~…~~ all of $\hat{\mathbb{C}}$.
(chaotic set).

Want explicit formula for f in this case:

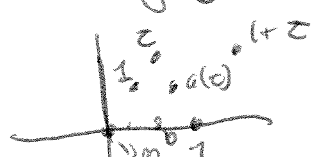
Facts: L is 4-to-1 (degree 4)

$\Rightarrow f$ is 4-to-1 (degree 4) $\Rightarrow$ exactly 4 poles (w/ mult), exactly 4 zeros (w/ mult).

Compose θ w/ mobius transformation to get $\wp : T \to \hat{\mathbb{C}}$ s.t.

$\wp(0) = \infty \qquad \wp(\tfrac{1}{2}) = 0$

$\wp(\tfrac{\tau}{2}) = 1 \qquad \wp(\tfrac{1+\tau}{2}) = a(\tau)$ (modular form)



$$\begin{tikzcd} z \arrow[r, mapsto] & 2z \\ T \arrow[r] \arrow[d, "\wp"'] & T \arrow[d, "\wp"] \\ \hat{\mathbb{C}} \arrow[r, "f"] & \hat{\mathbb{C}} \end{tikzcd}$$

$\rightsquigarrow$ $f(\wp(z)) = \wp(2\cdot z)$.

__Claim:__ $\text{poles}(f) = 0, 1, \infty, a$.

$$f(0) = f(\wp(0)) = \wp(2\cdot 0) = \wp(0) = \infty.$$
$$f(1) = f(\wp(\tfrac{\tau}{2})) = \wp(2\cdot\tfrac{\tau}{2}) = \wp(\tau) = \wp(0) = \infty.$$

etc.

__Claim:__ $\text{zeroes}(f) = \pm\sqrt{a}$

Consider $\tilde{h} : T \to T$ $\quad \tilde{h}(z) = z + 1/2 \quad (\Lambda)$.

$h : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \quad h(z) = \frac{a}{z}$.

__Claim:__ $h(\wp(z)) = \wp(\tilde{h}(z))$ __Exer.__ (similar trickery).

__w.t.s.__ $f(w) = 0 \Rightarrow h(w) = w$.

$w = \wp(z) \quad 0 = f(\wp(z)) = \wp(2\cdot z) \quad \Rightarrow 2\cdot z = \frac{1}{2} \quad (\Lambda)$

$\Rightarrow z = \frac{1}{4}, \frac{3}{4} (= -\frac{1}{4}) \quad (\Lambda)$.

$h(w) = h(\wp(\tfrac{1}{4})) = \cancel{\wp(\tfrac{a}{z})} = \wp(\tfrac{1}{4} + \tfrac{1}{2}) = \wp(\tfrac{-1}{4}) = \wp(\tfrac{1}{4}) = w.$

__Claim:__ $\pm\sqrt{a}$ ~~have~~ are zeroes of mult. 2.

__Exer:__ Similar trickery w/ $z \mapsto z + \frac{\tau}{2}$

$$\Rightarrow \quad f(w) = c \cdot \frac{(w^2 - a)^2}{w(w-1)(w-a)} \quad \Rightarrow f(w) = \frac{(w^2-a)^2}{4w(w-1)(w-a)}.$$

$f'(\infty) = 4$

$\Rightarrow c = \frac{1}{4}$

$$\begin{tikzcd} z \arrow[r, mapsto] & 2z \\ T \arrow[r] \arrow[d] & T \arrow[d] \\ \hat{\mathbb{C}} \arrow[r] & \hat{\mathbb{C}} \\ z^2 & (2z)^2 = 4z^2 \end{tikzcd}$$

Moral: If we vary τ $(\Lambda = \langle 1, \tau \rangle)$ (i.e. vary Torus)

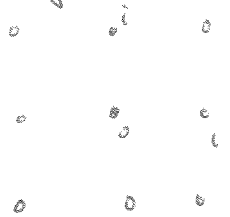
$\Rightarrow$ vary a, get 1-parameter family of Lattès maps.

All "flexible" Lattès maps look like

$$\begin{tikzcd} T \arrow[r, "{z \mapsto nz,\ nz + 1/2}"] \arrow[d] & T \arrow[d] \\ T/\{\pm z\} = \hat{\mathbb{C}} \arrow[r, "f"] & \hat{\mathbb{C}} = T/\{\pm z\} \end{tikzcd}$$

- deform Torus.

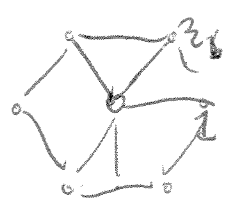
"Rigid" Lattès maps:



$\Lambda = \langle 1, i \rangle$ special since $i \cdot \Lambda \subseteq \Lambda$.

$$\begin{tikzcd} T \arrow[r, "z \mapsto \alpha z + \beta"] \arrow[d] & T \arrow[d] \\ \hat{\mathbb{C}} \arrow[r] & \hat{\mathbb{C}} = T/\{z \sim iz \sim -z \sim -iz\} \end{tikzcd}$$

Cannot "deform" torus since need to keep condition $i\Lambda \subseteq \Lambda$.



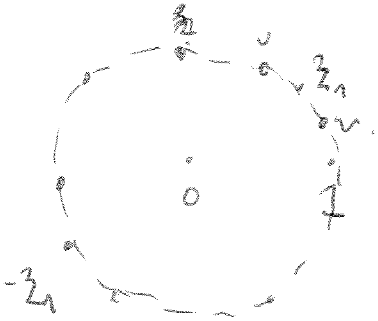
$\Lambda = \langle 1, \zeta_6 \rangle$ $\quad T \longmapsto \hat{\mathbb{C}} = T/\{z \sim \zeta_6^k z\}$ or $T/\{z \sim \zeta_3^k z\}$.

I.e. Modding out Torus by n^{th} roots of unity:

Claim: $n = 2, 3, 4, 6$.

Pf: Normalize Λ so smallest length of non-zero vector is 1.

~~Need to have $\zeta_n \cdot 1 \in \Lambda$ for all n~~



~~Assume n is even (since Λ closed under $z \mapsto -z$).~~

Claim: # ~~small~~ set smallest vectors is ≤ 6.

May assume # even since Λ closed under $z \mapsto -z$.

~~If # ≥ 8, pidgeonhole,~~

Circumference of circle is 2π so if had ≥ 8 smallest vectors, 2 of them v, w would be distance ≤ 1 apart on circle, $\Rightarrow$ $v - w$ smaller, contradiction

$\Rightarrow$ # smallest set of vectors $= 6$.