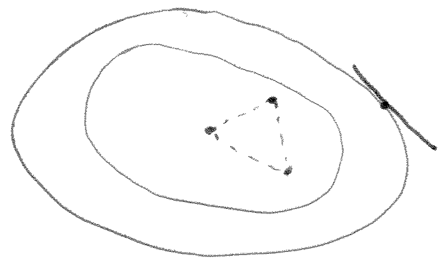
Convexity and (Wein)Stein Manifolds:

Regular Convexity: 3 definitions. $\Omega \subseteq \mathbb{R}^n$.

1) $\hat{K}_{\mathbb{R}^n} = \{ x \mid |f(x)| \leq \sup_K |f| \} \subset\subset \Omega$
f affine.
for all $K \subset\subset \Omega$



2) $\exists \, \phi$ s.t. $\Omega = \{ \phi \leq 0 \}$, $\left(\frac{\partial \phi}{\partial x_i \partial x_j} \right)$ pos. definite.

3) If line meets $\partial\Omega$ tangentially, line lies outside Ω.

$\Updownarrow$

$\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j} \right)$ pos definite on ~~$T_{\mathbb{R}}$~~ $T(\partial\Omega)$.

Holomorphic Convexity: 3 equiv. def's, $\Omega \subseteq \mathbb{C}^n$.

1) $\hat{K}_\Omega = \{ z \mid |f(z)| \leq \sup_K |f| \}$, $f \in A(\Omega)$

$K \subset\subset \Omega \Rightarrow \hat{K}_\Omega \subset\subset \Omega$ - holomorphic hull

2) $\Omega = \{ \varphi < 0 \}$. then

$\left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \right)$ strictly pos. definite.

3) Clx curve meets $\partial\Omega$ tangentially, then curve stays outside Ω.

$\left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} \right)$ pos definite on $T(\partial\Omega) \cap iT(\partial\Omega)$.

Stein Manifolds:

Def: Ω clx manifold is Stein if

1) $A(\Omega)$ seperates points
2) $A(\Omega)$ gives local coords for every point.
3) Ω holomorphically convex.

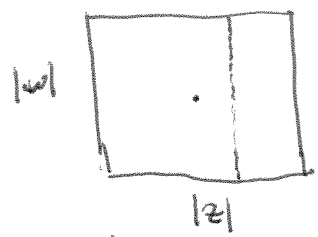
Examples: 0) $\mathbb{C}^n$
1) Analytic submanifolds of $\mathbb{C}^n$ } This is all of them

i.e. Stein manifolds ~ affine varieties.

Non-Example: $\mathbb{C}^2 \setminus \{0\}$

Lemma: Every $f \in A(\mathbb{C}^2 \setminus \{0\})$ extends to $\mathbb{C}^2$.

Pf: Write $f(z,w) = \sum_{n=-\infty}^{\infty} a_n(w) z^n$



Look @ set where $z \neq 0$.

$\Rightarrow a_n(w) \equiv 0$ if $n < 0$ $\Rightarrow$ no negative w terms.

+ ~~$a_n(w)$ cannot have negative w-terms.~~

~~since~~

By symmetry, no negative z-terms.

$\Rightarrow f(z,w)$ defined @ $(0,0)$. + extends!

~~So le~~

So let $K = S_\varepsilon(0)$ - sphere radius ε

~~Mak~~ Max principle $\Rightarrow \hat{K}_{\mathbb{C}^2 \setminus \{0\}} = B_\varepsilon(0) \setminus \{0,0\}$

- not compact!

Thm (Levi-problem)

Ω Stein iff no "extension problems" as above.

Rmk: $\exists$ exhausting strictly psh function

Morse w/ crit pts. of index $\leq n$.

J - convexity:

$\partial\bar\partial \varphi \geq 0 \iff d(d\phi \circ J) \geq 0$

<u>Def</u>: (V, J) is J-convex if $\exists \varphi : V \to \mathbb{R}$ exhausting J-convex function.

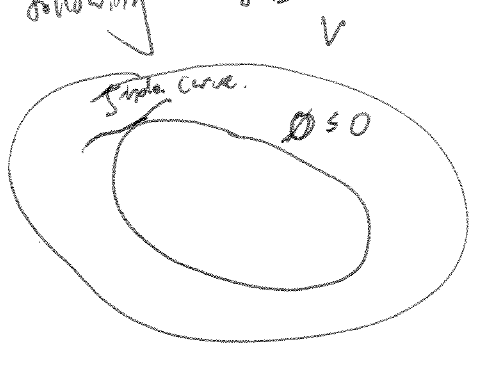
<u>Thm</u> (Eliashberg) (V, J) dim $2n > 4$, $\phi : V \to \mathbb{R}$ exhausting Morse function w/out crit pts of index $\geq n$. Then $\exists \tilde{J} \sim J$ integrable s.t. $(V, \tilde{J})$ is Stein.

~~Let~~

<u>Def</u>: <u>Weinstein Manifolds</u>: (V, ω, X, ϕ)

- ω symplectic for V.
- $\phi : V \to \mathbb{R}$ exhausting morse.
- X complete, $L_X \omega = \omega$, gradient-like for ϕ.
  - $\to$ $X = \text{grad}(f)$ near crit pts in coordinates.
    $\phi = -x_1^2 - \dots - x_e^2 + x_{e+1}^2 + \dots + x_n^2$.
  - $X \parallel \text{grad}(f)$ away from crit points.

<u>Q</u>: Are Weinstein manifolds convex in the following sense?



Topology of polynomially / rationally / holomorphically convex Domains.

<u>Def:</u> $\hat{K}_P = \{ z \in \mathbb{C}^n \mid |P(z)| \leq \max_K |P| \quad P \in \mathbb{C}[z] \}$

$\hat{K}_R = \{ z \in \mathbb{C}^n \mid |R(z)| \leq \max_K |R| \quad R \text{ rational, defined on } K \}$.

$\hat{K}_{A(\Omega)} = \{ z \in \Omega \mid |f(z)| \leq \max_K |f| \quad f \in A(\Omega) \}$.

<u>Def:</u> K poly convex if $\hat{K}_P = K$ $\Downarrow$

rat convex $\hat{K}_R = K$ $\Downarrow$) implications.

K holo. convex if $\forall\, U \supseteq K$

$\hat{K}_{A(\Omega)} \subset\subset U$

$\Rightarrow K = \bigcap_{U \supseteq K} \hat{K}_{A(\Omega)}$.

<u>Q:</u> What are topological restraints on ~~compact~~ convex domains?

<u>Thm</u> (Cieliebak - Eliashberg)

$W \subseteq \mathbb{C}^n$, $n \geq 3$ smoothly isotopic to rationally convex domain iff $\exists\, \phi$ defining morse function w/ no crit pts of index $\geq n$.

<u>Thm</u> (Cieliebak - Eliashberg)

$W \subseteq \mathbb{C}^n$, $n \geq 3$ — " — polynomially — "

— " — index $\geq n$