

## $\mathbb{CP}^n$ As A Kähler Manifold

$$\mathbb{CP}^n := \{ \text{lines (complex) in } \mathbb{C}^{n+1} \}$$
$$:= \mathbb{C}^{n+1} \setminus \{0\} / \mathbb{C}^*$$

$$:= \{ [z_0 : z_1 : \dots : z_n] \}$$

$$\text{where } [\lambda z_0 : \lambda z_1 : \dots : \lambda z_n] = [z_0 : z_1 : \dots : z_n]$$

$$U_j := \{ z_j \neq 0 \} \subseteq \mathbb{CP}^n$$

$$\phi_j: U_j \rightarrow \mathbb{C}^n$$

$$[z_0 : \dots : z_n] \mapsto \left( \frac{z_0}{z_j}, \dots, \frac{\hat{z_j}}{z_j}, \dots, \frac{z_n}{z_j} \right)$$

Def. Fubini - Study Form.

$$\omega_{FS} = \frac{i}{2} \partial \bar{\partial} f_j$$

$$f_j(z) := \log \left( \frac{\sum_{v=0}^n z_v \bar{z}_v}{z_j \bar{z}_j} \right)$$

Need to check:

- well-defined

- closed

- non-degenerate

- compatible

for closed:  $0 = d^2 = (\partial + \bar{\partial})^2 = \partial^2 + \bar{\partial}\partial + \partial\bar{\partial} + \bar{\partial}^2$

$$\Rightarrow \partial^2 = 0, \bar{\partial}^2 = 0 \quad \bar{\partial}\partial = -\partial\bar{\partial}$$

Well-defined:

$$U_0 \xrightarrow{g} U_n$$

$$z_1, \dots, z_n \quad w_0, \dots, w_{n-1}$$

$$w_0 = \frac{1}{z_n}$$

$$w_i = \frac{z_i}{z_n}$$

need to show:  $g^* (\partial \bar{\partial} \log (1 + \sum w_v \bar{w}_v)) = \partial \bar{\partial} \log (1 + \sum z_v \bar{z}_v)$

Since  $g$  holomorphic,

$$= \partial \bar{\partial} g^* \log (1 + \sum w_v \bar{w}_v)$$

$$= \partial \bar{\partial} \log \left( 1 + \frac{1}{z_n \bar{z}_n} + \sum \frac{z_v \bar{z}_v}{z_n \bar{z}_n} \right)$$

$$= \partial \bar{\partial} \left[ \log (1 + \sum z_v \bar{z}_v) - \log (z_n \bar{z}_n) \right]$$

$$\partial \bar{\partial} \log (z_n \bar{z}_n) = \partial \left( \frac{1}{z_n \bar{z}_n} z_n d\bar{z}_n \right) = \partial \left( \frac{1}{\bar{z}_n} d\bar{z}_n \right)$$

$$= 0$$

~~Also~~ Non-Degenerate / Compatible

Show  $\partial \bar{\partial} \log (1 + \sum z_v \bar{z}_v)$  is plurisubharmonic

$$\Leftrightarrow \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \Big|_{z=0} \log (1 + |z + \tau|^2) \geq 0$$

$$\frac{\partial}{\partial \tau} \left[ \frac{1}{1 + |z + \tau|^2} (z + \tau) \right]$$

$$\frac{-1}{(1 + |z + \tau|^2)^2} \frac{1}{(z + \tau)(\bar{z} + \bar{\tau})} + \frac{1}{1 + |z + \tau|^2}$$

$\tau = 0$

$$= \frac{-1}{(1 + |z|^2)^2} z \bar{z} + \frac{1}{1 + |z|^2}$$

$$= \frac{1 + |z|^2 - |z|^2}{(1 + |z|^2)^2} = \frac{1}{(1 + |z|^2)^2} \geq 0$$

## Lagrangian Submanifolds

$\mathbb{R}P^n \subset \mathbb{C}P^n$  is Lagrangian.

$$\partial \bar{\partial} \log(1+|z|^2) = \partial \left[ \frac{1}{1+|z|^2} z d\bar{z} \right]$$

$$= \frac{-1}{1+|z|^2} \bar{z} dz \wedge z d\bar{z} + \frac{1}{1+|z|^2} dz \wedge d\bar{z}$$

On  $\mathbb{R}P^n \subset \mathbb{C}P^n$ ,  $z = \bar{z} \Rightarrow dz = d\bar{z}$

so above expression is 0.

Clifford Torus is Lagrangian

$$\Pi^n := \{ [z_0 : \dots : z_n] \mid |z_0| = |z_1| = \dots = |z_n| \}$$

In affine chart, given by equations

$$z_i \bar{z}_i = 1$$

$$\Rightarrow \bar{z}_i dz_i + z_i d\bar{z}_i = 0 \Rightarrow 1^{\text{st}} \text{ term is zero}$$

$$\Rightarrow \cancel{d\bar{z}_i \wedge dz_i + dz_i \wedge d\bar{z}_i = 0}$$

Parametrize via  $z_j = e^{it}$   $\bar{z}_j = e^{-it}$

$$\Rightarrow dz_j = ie^{it} dt \quad d\bar{z}_j = -ie^{-it} dt$$

$$\Rightarrow dz_j \wedge d\bar{z}_j = 0.$$

Set  $z d\bar{z} = \sum z_j d\bar{z}_j$   
 $\bar{z} dz = \sum \bar{z}_j dz_j$   
 $dz \wedge d\bar{z} = \sum dz_j \wedge d\bar{z}_j$

Case  $n=1$ ,  $\mathbb{CP}^1 = S^2$

In coordinates  $z = x + iy$

$$\omega_{FS} = \frac{dx \wedge dy}{(1 + x^2 + y^2)^2}$$

$$\phi: S^2 \rightarrow \mathbb{CP}^1$$

- stereographic projection.

$$(x_1, x_2, x_3) \mapsto [1 + x_3 : x_1 + ix_2]$$

Claim:  $\phi^* \omega_{FS} = \frac{1}{4} d\text{vol}_{S^2} = \frac{1}{4} (x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2)$

Sketch: On  $U_1$ , the map is

$$(x_1, x_2, x_3) \mapsto \frac{1+x_3}{x_1+ix_2} = \frac{(1+x_3)x_1}{(x_1^2+x_2^2)} - i \frac{(1+x_3)x_2}{(x_1^2+x_2^2)}$$

$\begin{matrix} \parallel & \parallel \\ x & y \end{matrix}$

+ pull back.

$$\therefore \int_{\mathbb{CP}^1} \omega_{FS} = \frac{1}{4} \int_{S^2} d\text{vol}_{S^2} = \pi$$

Darboux Charts on  $\mathbb{CP}^n$ :

$$\psi_j([z_0 : \dots : z_n]) := \frac{|z_j|}{|z|} \left( \frac{z_0}{z_j}, \dots, \frac{z_n}{z_j} \right)$$

$$U_j \xrightarrow{\psi_j} \tilde{U}_j$$

$$z \mapsto \frac{1}{j+|z|^2} z$$

Can check that  $\psi_j^{-1*} \omega_{FS} = \omega_0$ .

Question from Book:

$$pr: \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{CP}^n$$

$$p_{FS} := \cancel{pr}^* p_{FS}^* \omega_{FS}.$$

$$\text{Claim: } \phi: \mathbb{C}^{n+1} \setminus \{0\} \rightarrow S^{2n+1}$$
$$z \mapsto \frac{z}{|z|}$$

$$\text{Then } \phi^* \omega_0 = p_{FS}^* \omega_{FS}, \quad \omega_0 = \sum_{k=0}^n dx_k \wedge dy_k$$

Does this imply  $p_{FS}$  non-degenerate?  
What about  $\omega_{FS}$ ?