Introduction

Signals are everywhere: stock prices, sound earthquakes, etc. We are interested in processing these signals to extract only the information we care about. Often, just plotting the signal of interest will not do it. In many cases, an analysis of how the signal is changing over time will also not help us. In such cases, we have to look toward other metrics. Fortunately, a number of very useful methods have been developed to help us in this case. As we will learn, instead of looking at the time-domain of a signal (which is likely what most of you have up until now), we can look at its frequency-domain.

Signals

For our purposes we will treat signals (sensor readings, for example) as mathematical functions. We can define a signal as a mathematical function

\[ f(\cdot) : t \rightarrow x \]

that maps an input \( t \) (scalar or vector) to a scalar \( x \). Our input does not necessarily have to be time, but in most cases for CEE575 it will be. For example:

\[ x(t) = A \sin(\omega t + \theta) \]

maps the scalar time input \( t \) to a sinusoidal output \( x(t) \). The above, in fact, is a special signal. Recall that a sinusoidal function can be fully described by its:

- \( A \) amplitude
- \( \omega \) frequency
- \( \theta \) phase (time shift)
Properties of periodic signals

A function \( x \) is said to have a period \( T \) if \( x(t + T) = x(t) \ \forall t \). Essentially, the function repeats itself every on interval \( T \). It can be proven that a well behaved \( T \)-periodic function \( x(t) \) can be approximated as sum of orthogonal basis functions \( \psi_k \), such that

\[
\hat{x}(t) = \sum_{k=0}^{\infty} a_k \psi_k(t)
\]

If you are interested in the details, in the above case \( \psi \) represent our set of orthogonal basis function of the transformation, where \( \alpha \) is a constant denoting their relative importance. The proof is, however, a bit beyond the scope of our course. In essence, just think of \( \hat{x}(t) \) as an approximation of \( x(t) \), that we obtain by adding together a number of other functions, each multiplied by some factor. The mazing thing is, that we can reconstruct \( x(t) \) entirely through this sum. We can even show that \( \hat{x}(t) \to x(t) \) as \( k \to \infty \). If we accept that \( \hat{x}(t) \) is a good enough approximation of our original signal, then we can just analyze \( \hat{x}(t) \), rather than \( x(t) \). Specifically, we will actually look at what happens to all of the \( \psi_k(t) \) functions. As we will see, this will have tremendous benefits.

This is how we end up with Fourier Series, which is just a very special case, where all of our \( \psi_k(t) \) will be sines and cosines.

**Fourier Series**

For a \( T \)-periodic function an approximation of our signal \( x(t) \) can be given by \( \hat{x}(t) \) is using a trigonometric relation involving **Fourier Series**, where

\[
\hat{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k \omega_0 t) + \sum_{k=0}^{\infty} b_k \sin(k \omega_0 t)
\]

where

\[
a_0 = \frac{1}{T} \int_{0}^{T} x(t) dt
\]

\[
a_k = \frac{2}{T} \int_{0}^{T} x(t) \cos(k \omega_0 t) dt
\]

\[
b_k = \frac{2}{T} \int_{0}^{T} x(t) \sin(k \omega_0 t) dt
\]

where

\[
\omega_0 = \frac{2\pi}{T}.
\]

The above is a very powerful relation, telling us that we can pretty much represent any function \( x(t) \) as a sum of sines and cosines. Please look at the
lecture slides to see how we can reconstruct even complex signals as a function of sines and cosines. Note, also, that due to symmetry properties we don’t have to necessarily integrate from \([0, T]\) above. Since sines and cosines have nice properties, it’s possible, and often easier to integrate across \([-\frac{T}{2}, \frac{T}{2}]\). It’s actually up to you what the interval us. Just make sure you integrate across an entire period \(T\).

Useful properties: Even and odd functions

If \(x(t)\) is an even function then \(x(t) = x(-t) \forall t\) and all of the sine coefficients \(b_k\) in the above approximation become zero. For example, take \(x(t) = \cos(t)\). The function is even, with a period of \(2\pi\). Then

\[
T = 2\pi \text{ (repeats every } 2\pi) \\
\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1
\]

\[
b_k = \frac{2}{2\pi} \int_0^{2\pi} \cos(t) \sin(k \cdot t) dt = \frac{1}{\pi} \int_0^{\pi} \cos(t) \sin(kt) dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \text{ even } \times \text{ odd}
\]

\[
= \frac{1}{\pi} \int_{-\pi}^{\pi} \text{ odd } = 0
\]

Above, we changed the integration interval from \([-\pi, \pi]\) to make the math easier (see tip in prior section). We then used some basic properties from pre-calc to determine that the inside of the integral will be odd. If that’s the case, the the integral to the left of zero will equal the one to the right. The total is thus equal to zero.

Likewise, it can be shown that if \(x(t)\) is odd [e.g. \(x(-t) = -x(t)\)], the cosine coefficients \(a_k\) become zero as well. Watching out for this can help us save time on math.

Example: Fourier decomposition

Consider the periodic signal given by a repeating square wave below. Find its Fourier series expansion for the periodic interval \(t \in [-1, 1]\).

\[
x(t) = \begin{cases} 
1 & -1 < t \leq 0 \\
-1 & 0 > t \geq 1 \\
x(t) & x(t + kT)
\end{cases}
\]

We know this is an odd function, with period \(T = 2\) so the \(a_k\) components will be zero. Then

\[
a_0 = \frac{1}{2} \int_{-1}^{1} x(t) dt = 0
\]
Figure 1: Example 1. Step function.

\[ a_k = 0 \text{ (since } x(t) \text{ is odd)} \]

\[ b_k = \frac{2}{k} \int_{-1}^{1} x(t) \sin(k\omega_0 t) \]

Note that we are integrating from \([-1, 1]\) instead of to make things easier, but you would get the same results in either case. Since \( w_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \)
then,

\[ b_k = \frac{2}{k} \int_{-1}^{0} 1 \cdot \sin(k\pi t) dt + \int_{0}^{1} (-1) \cdot \sin(k\pi t) dt \]

\[ = -\frac{1}{k\pi} \cos(k\pi t) \bigg|_{0}^{1} + \frac{1}{k\pi} \cos(k\pi t) \bigg|_{0}^{1} \]

\[ = \frac{1}{k\pi} (\cos(-k\pi) - 1 + \cos(k\pi) - 1) = \frac{1}{k\pi} (2\cos(k\pi) - 2) = -\frac{4\sin^2\left(\frac{k\pi}{2}\right)}{k\pi} \]

Then, through substitution \( \hat{x} \) becomes

\[ \hat{x}(t) = -\sum_{k=1}^{\infty} -\frac{4\sin^2\left(\frac{k\pi}{2}\right)}{k\pi} \sin(k\pi t) = -\sum_{k\text{ odd}}^{\infty} \frac{4}{k\pi} \sin(k\pi t) \]

In reality, we can not sum to infinity. The accuracy by which \( \hat{x}(t) \) approximates \( x(t) \) depends on how what values of \( k \) we use. The figure below shows the effect of increasing \( k \). As expected, larger values of \( k \) give us a better estimate of \( x(t) \).
Approximating $x(t)$ by partial sums of its Fourier series:

Note the **Gibbs phenomenon**: the Fourier series (over/under)shoots the actual value of $x(t)$ at points of discontinuity. In signal processing, this effect is also called **ringing**.

Figure 2: The effect of increasing the amount of time we sum over $k$ on $\hat{x}(t)$.
Source: http://maxim.ece.illinois.edu/teaching/fall08/lec8.pdf
Plotting Fourier Series Coefficients (amplitude and phase)

We can now study the coefficients $a_k$ and $b_k$ to learn more about our signal. A good place to start is to plot them. We can combine the $a_k$ and $b_k$ coefficients into a plottable amplitude $r_k$ and phase $\theta_k$ by using the well known trigonometric relation that a sine wave can be represented as a cosine wave with a phase shift.

The relationship between $r_k$ $a_k$ and $b_k$ can be derived by expanding the cosine with the phase shift (homework problem?), using trigonometric identities and comparing the results. For the $k^{th}$ term above we get

$$a_k \cos(\omega_0 t) + b_k \sin(\omega_0 t) = r_k \cos(\omega_0 t) \cos(\theta_k) + r_k \sin(\omega_0 t) \sin(\theta_k) = r_k \cos(\omega_0 t - \theta_k)$$

Where,

$$r_k = \sqrt{a_k^2 + b_k^2} \text{ and } \theta_k = \tan^{-1} \left( \frac{b_k}{a_k} \right)$$

Then,

$$x(t) = a_0 + \sum_{k=0}^{\infty} r_k \cos(\omega_0 t - \theta_k).$$

We can now plot these coefficients to obtain information about our signal. Please see the slides for examples.

A simpler representation

It is more common to express Fourier series using complex exponentials. You may remember these relations from your basic calculus or trigonometry course

$$\cos(\theta) = \frac{1}{2} (e^{j\theta} + e^{-j\theta})$$

$$\sin(\theta) = \frac{1}{2j} (e^{j\theta} - e^{-j\theta})$$

where $j = \sqrt{-1}$. From these equations we can derive an even simpler (easier to work with) expression for our Fourier series, by showing that

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

where

$$c_k = \frac{1}{T} \int_{0}^{T} x(t) e^{-jk\omega_0} dt$$

Furthermore, the complex conjugate of an imaginary number $z = a + jb$ is given by $\bar{z}$ where

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\[ \bar{z} = a - ib \]

For a real-valued signal \( x(t) \) it can be shown that \( c_k \) and \( c_{-k} \) are complex conjugates (left for homework). Consequently, the above relation can be simplified as follows

\[ x(t) = c_0 + 2 \cdot \sum_{k=1}^{\infty} c_k e^{j w_0 t} \]

**Interpretation**

In fact, there is a very close relationship between \( c_k \) and the amplitude \( r_k \) and phase \( \theta_k \) components mentioned earlier. \( c_k \) is a complex number that can be written in polar form

\[ c_k = Ae^{j\theta} \]

where \( A \) is the amplitude of the complex number, and \( \theta \) is the angle in the complex plane. It can be shown that

\[ A = r_k = |c_k| \]

and

\[ \theta = \theta_k = \angle c_k \]

Therefore the trigonometric form for Fourier series embeds information about both the amplitude and phase of our frequency spectrum!