Winter 2019 LoG(M) Project: Lower Dimensional Flag Triangulated Manifolds

Christin Bibby & Hailun Zheng

January 1, 2019

Abstract

In this project, we will explore some open problems on the face enumeration of lower dimensional flag triangulated manifolds. A triangulation is a simplicial complex which, roughly speaking, describes how to build a space (such as a sphere or a torus) out of triangles. Understanding the combinatorics of triangulations has always been an important research aspect of geometric combinatorics. A fundamental combinatorial invariant of a triangulated manifold is its f-vector, which encodes the number of faces in each dimension. Not surprisingly, the topology of the manifold affects the f-vectors of its triangulations.

The 1-skeleton of a simplicial complex can be viewed as a graph, and a simplicial complex (or triangulation) is said to be flag if the faces correspond to cliques (complete subgraphs) in this graph. Examples of flag complexes include barycentric subdivisions of simplicial complexes and of Coxeter complexes. The additional structure of flag complexes makes their combinatorics quite different from those of general simplicial complexes. For example, an observation by Gromov leads to an elegant combinatorial interpretation of the celebrated Hopf conjecture from differential geometry, known as the Charney-Davis conjecture. This conjecture states that for odd-dimensional flag simplicial spheres, a certain linear combination of the face numbers is nonnegative.

This project will start by exploring (minimal) flag triangulations of surfaces: It is already known that an octahedron gives a flag triangulation of the sphere with the minimum number of vertices, but less is known for other surfaces (such as a torus). Once comfortable with flag surfaces, we will then explore the relationship between the face numbers and topological invariants of flag triangulated 3-manifolds, in the spirit of the Charney-Davis conjecture and related works.

Pre-requisites are Math 217 or equivalent. Experience in Math 490 and/or Math 465, along with basic coding skills (Sage or Python) are strongly suggested.

Students will be expected to keep track of hours worked on timesheets.

1 Flag triangulations and f-vectors

Let Δ be a simplicial complex (or triangulation) of dimension $d-1$. The f-vector of Δ is defined to be $(f_0, \ldots, f_{d-1})$, where f_i is the number of i-dimensional faces. The numbers $f_0, \ldots, f_{d-1}$ are often called the face numbers of Δ. We will see in (2.1) below a relationship between the f-vector of a triangulation and the Euler characteristic of the underlying manifold.

The 1-skeleton, i.e. the 0- and 1-simplices, of Δ may be viewed as a graph with f_0 vertices and f_1 edges. The triangulation Δ is said to be *flag* if all the faces correspond to cliques (complete subgraphs) in this graph; equivalently, a triangulation is flag if all of its minimal non-faces has cardinality two. Flagness is of a particular interest to geometers. For instance, Gromov [4] noticed that when a piecewise Euclidean cubical complex is endowed with a certain metric, then the property that the complex has non-positive curvature is equivalent to the condition that every vertex link in the complex is flag, see Section 2 below. Flag complexes are also of interest in algebra as their Stanley-Reisner ideals are quadratic, and hence the associated rings are Koszul.

2 From Hopf conjecture to Charney-Davis conjecture

A complete metric space Y is a geodesic space (or "length space") if, given points $y_1, y_2 \in Y$, there is a path γ from y_1 to y_2 with $\ell(\gamma) = d(y_1, y_2)$. Such a distance minimizing path is called a geodesic.

Given a piecewise Euclidean cell complex Δ, assign each edge in Δ with length $\pi/2$. Given any points $y_1, y_2 \in \Delta$, define $d(y_1, y_2) = \inf\{\ell(\gamma)\}$, where the infimum is over all paths γ from y_1 to y_2 of finite length and ℓ measures the length of the path. In this way Δ is a geodesic space.

Lemma 2.1 (Gromov [4])**.** *The vertex links in any non-positively curved piecewise Euclidean cell complex are flag.*

We recall the celebrated Hopf conjecture.

Conjecture 2.2 (Hopf)**.** *Let M^{2n} be a closed non-positively curved $2n$-dimensional Riemannian manifold. Then $(-1)^n \chi(M^{2n}) \geq 0$.*

When $n = 1$, the conjecture follows from Gauss-Bonnet Theorem. The case $n = 2$ was proved by Chern and attributed to Milnor. The conjecture remains open for $n > 2$.

In 1995, Charney and Davis [1], by using Gromov's lemma and the fact that

$$\chi(\Delta) = 1 + \sum_i (-1/2)^{i+1} f_i(\Delta), \tag{2.1}$$

proposed a discrete version of the Hopf Conjecture as follows. Recall that $f_i(\Delta)$ denotes the number of i-dimensional faces in Δ.

Conjecture 2.3 (Charney-Davis)**.** *Let Δ be a flag triangulated sphere of dimension $2n-1$. Then*

$$(-1)^n\left(1-\frac{1}{2}f_0(\Delta)+\frac{1}{4}f_1(\Delta)+\cdots+\left(\frac{1}{2}\right)^{2n}f_{2n-1}(\Delta)\right)\geq 0.$$

The conjecture is proved for $n=2$ in [2] and is open in general.

3 Lower bound theorems for triangulated manifolds

As we see from Section 2, the Hopf conjecture has a combinatorial analog, which says that a certain linear combination of face numbers is nonnegative. This reminds us of many other lower bound results on various classes of triangulated manifolds. In the following, we let β_i be the rank of the ith homology group of Δ.

Theorem 3.1 (Novik and Swartz [9], Murai [7])**.** *Let Δ be a connected triangulated $(d-1)$-manifold. Then*

$$g_2(\Delta):=f_1(\Delta)-df_0(\Delta)+\binom{d+1}{2}\geq\binom{d+1}{2}\beta_1(\Delta).$$

In particular, if Δ is a triangulated sphere, then $g_2(\Delta)\geq 0$. Furthermore, equality holds if and only if Δ is a stacked manifold.

A $(d-1)$-dimensional simplicial complex Δ is *balanced* if there is a proper vertex coloring that uses at most d colors; that is, a map $\kappa: V(\Delta)\to\{1,\ldots,d\}$ such that for any edge $\{a,b\}\in\Delta$, $\kappa(a)\neq\kappa(b)$. The following is the balanced lower bound theorem for manifolds.

Theorem 3.2 (Juhnke-Kubitzke, Murai, Novik, Sawaske [5])**.** *Let Δ be a connected balanced triangulated $(n-1)$-manifold. Then*

$$\bar{g}_2(\Delta):=2f_1(\Delta)-3(d-1)f_0(\Delta)+4\binom{d}{2}\geq 4\binom{d}{2}\beta_1(\Delta).$$

In particular, if Δ is a balanced triangulated sphere, then $\bar{g}_2(\Delta)\geq 0$. Furthermore, equality holds if and only if Δ is a balanced stacked manifold.

In 2005 Gal [3] introduced the γ-vector for flag triangulated spheres. Using the γ-vector, the Charney-Davis conjecture in dimension 3 (now a theorem) can be rephrased as

$$\gamma_2(\Delta):=f_1(\Delta)-5f_0(\Delta)+16\geq 0.$$

He further proposed the following conjecture.

Conjecture 3.3 (Gal [3])**.** *Let Δ be a flag triangulated $(d-1)$-sphere. Then $\gamma_i(\Delta)\geq 0$ for all $1\leq i\leq\frac{d}{2}$.*

4 List of problems

In this project we will focus on 2- and 3-dimensional flag triangulated manifolds. With the aid of computer programs, many vertex-minimal triangulation of surfaces and 3-manifolds were found, see the manifold [6]. Also it is easy to see that the octahedral $(d-1)$-spheres give the unique vertex-minimal triangulation of $(d-1)$-spheres for all d. However, less is known about other flag manifolds. For example, the minimal triangulation of the torus $\mathbb{S}^1 \times \mathbb{S}^1$ has 7 vertices; furthermore, it is 2-neighborly and hence far from being flag. It is not even known how many vertices are required to form a flag triangulation of $\mathbb{S}^1 \times \mathbb{S}^1$.

Problem 4.1. *Find the minimal flag triangulation of the surfaces.*

This problem will involve the following steps:

1. Understand why an octahedron is the minimal flag triangulation of the 2-sphere.
2. Show that a flag triangulation of $\mathbb{R}\mathrm{P}^2$ requires at least six vertices.
3. Explore flag triangulations of the torus $\mathbb{S}^1 \times \mathbb{S}^1$.

On the other hand, it is even more interesting to explore flag 3-manifolds. Based on the discussion in Section 2, it is natural to guess that a plausible lower bound conjecture for flag 3-manifolds should be related to both the γ-vector and the first Betti number (or perhaps other topological invariants).

Problem 4.2. *Explore the relation between f_1, f_0 and β_1 (or π_1) for flag 3-manifolds.*

References

[1] R. Charney and M. Davis: The Euler characteristic of a non-positively curved, piecewise Euclidean manifold, *Pacific J. Math.* 171(1995), 117–137.

[2] M. Davis and B. Okun: Vanishing theorems and conjectures for the ℓ^2-homology of right-angled Coxeter groups, *Geom. Topol.* 5(2001), 7–74.

[3] Ś. R. Gal: Real root conjecture fails for five and higher dimensional spheres, *Discrete Comput. Geom.* 34(2005), 269–284.

[4] M. Gromov: Hyperbolic groups in "Essays in Group Theory", M.S.R.I Publ. 8, Springer-Verlag, New York and Berlin, 1987, 75–264.

[5] M. Juhnke-Kubitzke, S. Murai, I. Novik and C. Sawaske: A generalized lower bound theorem for balanced manifolds, *Mathematische Zeitschrift*, 289(2018), 921–942.

[6] `https://page.math.tu-berlin.de/~lutz/stellar/`

[7] S. Murai: Tight combinatorial manifolds and graded Betti numbers, Collect. Math. 66(2015), 367–386.

[8] F. Lutz and E. Nevo: Stellar theory for flag complexes, Math. Scand. 118(2016), 70–82.

[9] I. Novik and E. Swartz: Socles of Buchsbaum modules, complexes and posets, Adv. Math. 222(2009), 2059-2084.