LECTURE VIII: PERFECTIONS IN MIXED CHARACTERISTIC

Fix a perfect prism (A, I) and a p-complete A/I-algebra R. The goal of this lecture is to use prismatic cohomology to construct a canonical "perfectoidization" R_{perfd} of R. Let us warn the reader right away that unless $I = (p)$, the object R_{perfd} is not a perfectoid ring, but rather is a derived analogue. Nevertheless, in some important special cases, there is no derived structure, so R_{perfd} is an ordinary perfectoid ring. Using the Hodge-Tate comparison to control R_{perfd} in such cases, we reprove (and slightly improve) a fundamental theorem of André; using this, we prove that Zariski closed subsets of perfectoid spaces are strongly Zariski closed. In the next lecture, we shall utilize this construction in the proof of the étale comparison theorem for prismatic cohomology.

1. Perfection in characteristic p

In this section, we fix a perfect ground field k of characteristic p. Let us recall the classical notion of perfection for a k-algebra.

Definition 1.1 (Perfection)**.** For a k-algebra R, the *perfection* R_{perf} of R is defined as the direct limit

$$R_{\mathrm{perf}} := \mathrm{colim}\left(R \xrightarrow{\phi} R \xrightarrow{\phi} R \to \ldots\right),$$

where all the transition maps are the Frobenius on R. The R-algebra structure on R_{perf} is defined via the map $R \to R_{\mathrm{perf}}$ coming from the first term of the direct limit. (In the literature, R_{perf} is sometimes called R^{1/p^∞}.)

Example 1.2. If $R = k[x] = \bigoplus_{i \in \mathbf{N}} k \cdot x^i$, then $R_{\mathrm{perf}} = k[x^{1/p^\infty}] = \bigoplus_{i \in \mathbf{N}[1/p]} k \cdot x^i$. More generally, a similar description applies to the monoid algebra $R = k[M]$ for any commutative monoid M.

Remark 1.3. The direct limit diagram appearing in Definition 1.1 is not a diagram of k-algebras as the Frobenius on R is not k-linear, but only semilinear with respect to the Frobenius. To make it clear that the construction $R \mapsto R_{\mathrm{perf}}$ is a functor on k-algebras, one could instead define R_{perf} as the direct limit of the diagram

$$R \xrightarrow{\phi} \phi_* R \xrightarrow{\phi} \phi_*^2 R \to \ldots,$$

where all maps are now k-linear. To avoid an excessive proliferation of Frobenius twists, we do not do so above or in the sequel.

The aforementioned map $R \to R_{\mathrm{perf}}$ is the universal map from R into a perfect k-algebra. For future reference, we note that the relative Frobenius map $R^{(1)} \to R$ is inverted by the perfection, i.e., $(R^{(1)})_{\mathrm{perf}} \simeq R_{\mathrm{perf}}$. More generally, one can show that if $R \to S$ is a map of k-algebras that induces a universal homeomorphism on $\mathrm{Spec}(-)$, then $R_{\mathrm{perf}} \simeq S_{\mathrm{perf}}$.

Remark 1.4. Let R be a finite type k-algebra. Then a foundational theorem of Kunz asserts that the Frobenius on R is flat if and only if R is smooth over k. It is easy to see that the flatness of the Frobenius $R \to R$ is equivalent to the flatness of the map $R \to R_{\mathrm{perf}}$. In other words, the homological properties of $R \to R_{\mathrm{perf}}$ reflect the singularities of R.

With an eye towards mixed characteristic, let us give an alternative construction of the perfection that does not directly use the Frobenius on R. For this, recall that we have constructed (Definition VII.3.1) the derived de Rham cohomology functor $\mathrm{dR}_{-/k}$ as a left derived functor of $\Omega^*_{-/k}$ on polynomial k-algebras. By functoriality, the Frobenius on R induces a ϕ_k-semilinear endomorphism $\phi_R : \mathrm{dR}_{R/k} \to \mathrm{dR}_{R/k}$. Then one can reconstruct R_{perf} as the "perfection of $\mathrm{dR}_{R/k}$" as follows:

Proposition 1.5 (Perfections via derived de Rham cohomology)**.** *Let R be an k-algebra. The direct limit*

$$\mathrm{dR}_{R/k,\mathrm{perf}} := \mathrm{colim}\left(\mathrm{dR}_{R/k} \xrightarrow{\phi_R} \mathrm{dR}_{R/k} \xrightarrow{\phi_R} \mathrm{dR}_{R/k} \to ...\right),$$

identifies with R_{perf} via the canonical map $\mathrm{dR}_{R/k} \to R$ on the terms of the direct limit.

Proof. By general nonsense with left derived functors, it suffices to prove the statement when R is a smooth k-algebra (or even a polynomial k-algebra). In this case, we must show that the map $\Omega^*_{R/k} \to R$ obtained by projecting onto the 0-th term becomes an isomorphism when we apply the perfection operation to both sides. Unwinding definition, it suffices to show that the direct limit

$$\mathrm{colim}\left(\Omega^i_{R/k} \xrightarrow{\phi_R} \Omega^i_{R/k} \xrightarrow{\phi_R} \Omega^i_{R/k} \to ...\right)$$

vanishes for $i > 0$. But this is clear as Frobenius kills differential forms: in degree 1 this follows from $\phi_R(xdy) = x^p d(y^p) = x^p p y^{p-1} dy = 0$, and the higher degree case is similar. $\square$

We now reformulate the above statement in terms of derived prismatic cohomology (Definition IV.4.1). Write $(A,(p))$ for the perfect prism corresponding to k under Theorem IV.2.3, so $A \simeq W(k)$ explicitly. For any k-algebra R, we have defined the derived prismatic cohomology $\Delta_{R/A} \in D(A)$. By construction, this is a commutative algebra object of $D(A)$ that comes equipped with a ϕ_A-semilinear endomorphism $\phi_R : \Delta_{R/A} \to \Delta_{R/A}$. In particular, we have an induced Frobenius on the derived Hodge-Tate cohomology $\overline{\Delta}_{R/A} := \Delta_{R/A} \otimes^L_A A/(p)$ as well. In these terms, we can reformulate Proposition 1.5 as the following result reconstructing R_{perf} from the pair $(\overline{\Delta}_{R/A}, \phi_R)$.

Proposition 1.6 (Perfections via Hodge-Tate cohomology in characteristic p)**.** *Let R be a k-algebra and let $A = W(k)$ as above. The direct limit*

$$\overline{\Delta}_{R/A,\mathrm{perf}} := \mathrm{colim}\left(\overline{\Delta}_{R/A} \xrightarrow{\phi} \overline{\Delta}_{R/A} \xrightarrow{\phi} \overline{\Delta}_{R/A} ...\right),$$

identifies with R_{perf} via the canonical map $R \to \overline{\Delta}_{R/A}$ on the terms of the direct limit.

Proof. The proof is entirely analogous to the proof of Proposition 1.5 using the Hodge-Tate filtration on $\overline{\Delta}_{R/A}$ instead of the Hodge filtration used in Proposition 1.5. More precisely, to run this argument, one needs to know that the map $\mathrm{gr}^{HT}_i(\phi_R) : \mathrm{gr}^{HT}_i(\overline{\Delta}_{R/A}) \to \mathrm{gr}^{HT}_i(\overline{\Delta}_{R/A})$ induced by the Frobenius ϕ_R on $\overline{\Delta}_{R/A}$ coincides with the map $\Omega^i_{R/k} \to \Omega^i_{R/k}$ induced by the Frobenius on R under the identification $\mathrm{gr}^{HT}_i(\overline{\Delta}_{R/A}) \simeq \Omega^i_{R/k}$ of Proposition VII.4.2. In other words, we must show that that $\mathrm{gr}^{HT}_i(\phi_R)$ is 0 if $i > 0$, and coincides with the Frobenius on R for $i = 0$ under the Hodge-Tate comparison isomorphism $R \simeq \mathrm{gr}^{HT}_0(\overline{\Delta}_{R/A})$. This compatibility can be seen directly for $R = k[x]$ via an explicit cosimplicial argument (compare with Lemma V.5.4), and the rest can be deduced by functoriality; we leave this check as an exercise for the reader. $\square$

In mixed characteristic, the Hodge-Tate complex will *not* carry a Frobenius. It is therefore convenient to reformulate the preceding statement in terms of prismatic cohomology.

Corollary 1.7 (Perfections of prismatic cohomology in characteristic p)**.** *Let R be a k-algebra, and let $A = W(k)$ as above. Then the p-completed direct limit*

$$\Delta_{R/A,\mathrm{perf}} := \mathrm{colim}\left(\Delta_{R/A} \xrightarrow{\phi_R} \Delta_{R/A} \xrightarrow{\phi_R} \Delta_{R/A} \to ...\right)^{\wedge} \in D(\mathbf{Z}_p)$$

identifies with $W(R_{\mathrm{perf}})$ as a $W(k)$-algebra. In particular, it is concentrated in degree 0.

2. Perfection in mixed characteristic

Fix a perfect prism (A, I). Our goal is to explain how to construct a canonical "perfectoidization" for A/I-algebras R drawing inspiration from Corollary 1.7.

Notation 2.1. Recall that $I = (d)$ is generated by a distinguished element d (Lemma III.3.5). We write $D_{comp}(A)$ for the full subcategory of $D(A)$ spanned by derived (p, I)-complete complexes and refer to it as the (p, I)-complete derived category of A if there is no potential for confusion; similarly, for any A/I-algebra R, we write $D_{comp}(R)$ for the p-complete derived category of R.

Definition 2.2 (Perfectoidizations)**.** For any p-complete A/I-algebra R, we define the following:

(1) The *perfection* $\Delta_{R/A,\mathrm{perf}}$ is defined as the colimit

$$\Delta_{R/A,\mathrm{perf}} := \mathrm{colim}\left(\Delta_{R/A} \xrightarrow{\phi_R} \Delta_{R/A} \xrightarrow{\phi_R} \Delta_{R/A} \to \ldots\right)^\wedge \in D_{comp}(A),$$

where $(-)^\wedge$ denotes derived (p, I)-completion, and the A-linear structure is defined using the canonical map $\Delta_{R/A} \to \Delta_{R/A,\mathrm{perf}}$ coming from the first term in the colimit above.

(2) The *perfectoidization* R_{perfd} is defined

$$R_{\mathrm{perfd}} := \Delta_{R/A,\mathrm{perf}} \otimes^L_A A/I \in D_{comp}(R),$$

where the R-linear structure is defined via the canonical map $R \to \overline{\Delta}_{R/A} \to \Delta_{R/A,\mathrm{perf}} \otimes^L_A A/I$. (The notation suggests that R_{perfd} is independent of the base perfect prism (A, I), and this will be checked in Lemma 2.6 below.)

By general nonsense, $\Delta_{R/A,\mathrm{perf}} \in D_{comp}(A)$ and $R_{\mathrm{perfd}} \in D_{comp}(R)$ are commutative algebra objects. Moreover, the Frobenius ϕ_R of $\Delta_{R/A}$ induces an automorphism of $\Delta_{R/A,\mathrm{perf}}$ also called ϕ_R. Let us give some examples.

Example 2.3. Fix a perfect prism (A, I) as above with R being a p-complete A/I-algebra.

(1) (Crystalline prisms) When $I = (p)$, it follows from Corollary 1.7 that $R_{\mathrm{perfd}} \simeq R_{\mathrm{perf}}$ and $\Delta_{R/A,\mathrm{perf}} \simeq W(R_{\mathrm{perf}})$; in particular, both are concentrated in degree 0.

(2) (Perfectoid rings) Assume R is itself a perfectoid ring. Then $\Delta_{R/A} \simeq A_{\mathrm{inf}}(R)$ is already perfect (see proof in next paragraph), and hence $\Delta_{R/A,\mathrm{perf}} \simeq A_{\mathrm{inf}}(R)$ with $R \simeq R_{\mathrm{perfd}}$. In particular, they are both concentrated in degree 0.

To see that $\Delta_{R/A} \simeq A_{\mathrm{inf}}(R)$, note that there is a natural map $\Delta_{R/A} \to A_{\mathrm{inf}}(R)$ coming from the universal property of $\Delta_{R/A}$ as $(R \to A_{\mathrm{inf}}(R)/IA_{\mathrm{inf}}(R) \leftarrow A_{\mathrm{inf}}(R))$ gives an object of $(R/A)_{\Delta}$ (since $IA_{\mathrm{inf}}(R) = \ker(\theta_R)$ via Theorem V.2.3). To check that $\Delta_{R/A} \to A_{\mathrm{inf}}(R)$ is an isomorphism, it suffices to do so after applying $- \otimes^L_A A/I$ by derived Nakayama. We are then reduced to checking that $\overline{\Delta}_{R/A} \simeq R$, which follows from considerations of the Hodge-Tate filtration (Proposition VII.4.2) since $L_{R/(A/I)}$ and its exterior powers vanish after derived p-completion as R is perfectoid (Exercise VII.2.2)

(3) (The torus) Assume (A, I) is the perfection of $(\mathbf{Z}_p[\![q-1]\!], ([p]_q))$ (see Lemma IV.1.3), so A is the $(p, [p]_q)$-completion of $\mathbf{Z}_p[q^{1/p^\infty}]$. Say $R = A/I[x^{\pm 1}]^\wedge$ is the p-adic completion of a Laurent polynomial ring. In this case, we have $H^1(\Delta_{R/A,\mathrm{perf}}) \neq 0$ and $H^1(R_{\mathrm{perfd}}) \neq 0$. Indeed, as we shall explain later, $\Delta_{R/A}$ is computed by a q-de Rham complex as in Remark I.3.4 (f), and hence $\Delta_{R/A,\mathrm{perf}}$ is computed by its perfection. Explicitly, this turns out to be the the $(p, [p]_q)$-completion of the 2-term complex

$$A[x^{\pm 1/p^\infty}] \xrightarrow{\gamma - \mathrm{id}} JA[x^{\pm 1/p^\infty}],$$

where $J = (\cup_n (q^{1/p^n} - 1))^\wedge$ is the kernel of the natural map $A \to \mathbf{Z}_p$ defined by $q^{1/p^n} \mapsto 1$ for all n, and γ is the map determined by $\gamma(x^i) = q^i x$ for $i \in \mathbf{Z}[1/p]$. In particular, one checks that the element $(q-1)\cdot 1$ in the degree 1 term above is not a boundary even modulo $[p]_q$, so $H^1(\Delta_{R/A,\mathrm{perf}}) \neq 0$ and $H^1(R_{\mathrm{perfd}}) \neq 0$.

Remark 2.4. Example 2.3 (3) is somewhat disconcerting at first glance: the perfectoidization R_{perfd} of a (very nice) A/I-algebra R is not concentrated in degree 0. This is in stark contrast to what happens in characteristic p situation (i.e., when $I = (p)$). However, we shall see later that the presence of positive degree cohomology of R_{perfd} (and thus $\Delta_{R/A,\mathrm{perf}}$) is a feature as it carries geometric meaning: $H^i(\Delta_{R/A,\mathrm{perf}})$ can be used to recover $H^i(\mathrm{Spec}(R[1/p]), \mathbf{F}_p)$.

Remark 2.5 (The structure of the perfection)**.** The Frobenius ϕ_R endows $\Delta_{R/A,\mathrm{perf}}$ with the structure of a "derived perfect δ-ring". We do not spell out the definition here and simply give a plausibility argument: the pair $(\Delta_{R/A}, \phi_R)$ is computed via limits and colimits[1] over the simplex category Δ applied to some diagrams of δ-rings, so it acquires a derived δ-structure (encoded by a map $\Delta_{R/A} \to W_2(\Delta_{R/A})$) by functoriality, and thus its perfection $\Delta_{R/A,\mathrm{perf}}$ becomes a derived perfect δ-ring. Concrete consequences include:

(1) (Coconnectivity of the perfection) $\Delta_{R/A,\mathrm{perf}}$ always lies in $D^{\geq 0}$, i.e. we have $H^i(\Delta_{R/A,\mathrm{perf}}) = 0$ for $i < 0$. Indeed, any commutative algebra object in $\mathcal{D}(\mathbf{F}_p)$ obtained from a diagram of $\mathbf{F}_p$-algebras via limits and colimits (such as $\Delta_{R/A,\mathrm{perf}}/p$) carries a natural Frobenius endomorphism by functoriality, and this endomorphism is always 0 on negative cohomology groups (see [2, §11] for an explicit argument for simplicial commutative rings, and [4, Remark 2.2.7] for a general argument). If the algebra also happens to be perfect (i.e., have a bijective Frobenius), it follows that the negative cohomology groups must vanish.

(2) (The semiperfectoid case) If $\Delta_{R/A,\mathrm{perf}}$ lies in $D^{\leq 0}$, then it is concentrated in degree 0 by (1), and is thus a perfect (p, I)-complete δ-ring. Consequently, we get a perfect prism $(\Delta_{R/A,\mathrm{perf}}, I\Delta_{R/A,\mathrm{perf}})$ over (A, I). In this case, R_{perfd} is a perfectoid ring, and the map $R \to R_{\mathrm{perfd}}$ is the universal map to a perfectoid ring thanks to Example 2.3 (2).

An important example where the hypothesis in (2) is satisfied is given by semiperfectoid rings (which also explains the name chosen above), see Corollary 3.2 below.

The following independence result was promised earlier.

Lemma 2.6 (Independence of the base)**.** *Let $(A, I) \to (B, J)$ be a map of perfect prisms, and let S be a p-complete B/J-algebra. Then the natural map gives an isomorphism $\Delta_{S/A} \simeq \Delta_{S/B}$ and an isomorphism $\Delta_{S/A,\mathrm{perf}} \simeq \Delta_{S/B,\mathrm{perf}}$.*

Consequently, the perfectoidization S_{perfd} of a p-complete ring S can be computed using any perfect prism (A, I) such that S admits an A/I-algebra structure (assuming one exists).

Proof. By derived Nakayama, it suffices to check $\overline{\Delta}_{S/A} \simeq \overline{\Delta}_{S/B}$. This follows by considerations of the Hodge-Tate filtration and the fact that $L_{(B/J)/(A/I)}$ vanishes after p-completion as A/I and B/J are both perfectoid (Exercise VII.2.2). ☐

The following stability property will be important later.

Lemma 2.7 (Base change compatibility)**.** *The functor $R \mapsto R_{\mathrm{perfd}}$ commutes with (p, I)-completely faithfully flat base change on the perfect prism (A, I).*

[1] The limit is from the description of prismatic cohomology of a smooth A/I-algebra via Cech-Alexander complexes (Construction V.5.3), while the colimit comes from the construction of left derived functors (Remark VII.1.3).

Proof. We only explain the proof when R is bounded, i.e., $R[p^\infty] = R[p^n]$ for $n \gg 0$. Say $(A, I) \to (B, IB)$ is a faithfully flat map of perfect prisms. Write $S := R\widehat{\otimes}^L_{A/I} B/IB$. Then S is p-completely R-flat and thus concentrated in degree 0 as R is bounded. We must show that $R_{\mathrm{perfd}}\widehat{\otimes}^L_{A/I} B/IB \simeq S_{\mathrm{perfd}}$. By compatibility with filtered colimits, it is enough to show that $\overline{\Delta}_{R/A} \otimes^L_{A/I} B/IB \simeq \overline{\Delta}_{S/B}$. This follows by comparing the Hodge-Tate filtration on both sides, and using an analogous base change compatibility for the cotangent complex and its exterior powers. □

3. André's flatness lemma and consequences

Let us use prismatic cohomology to answer certain algebraic questions about perfectoid rings. We begin with a result showing roughly that any element of any perfectoid ring admits all p-power roots fpqc locally in the category of perfectoid rings.

Theorem 3.1 (André)**.** *Let R be a perfectoid ring. For any set $\{f_s \in R\}_{s\in S}$ of elements of R, there exists a p-completely faithfully flat map $R \to R_\infty$ of perfectoid rings such that each f_s admits a compatible system of p-power roots in R_∞. In other words, the map $\sharp : R^\flat \to R$ (Definition IV.2.8) is surjective locally for the p-completely flat topology.*

The almost variant of this result was proven in [1] for a specific perfectoid ring R using the theory of perfectoid spaces [5] (especially the tilting correspondence); see [3] for an exposition of André's proof in approximately the above generality.

Proof. As p-completely faithfully flat maps are stable under filtered colimits and coproducts, it suffices to explain the proof for a single element, i.e., when $\#S = 1$. Write $f = f_s$. Let $(A, I) = (A_{\mathrm{inf}}(R), \ker(\theta_R))$ be the perfect prism corresponding to R. Let S be the p-adic completion of $R[x^{1/p^\infty}]/(x - f)$, so $R \to S$ is p-completely faithfully flat, and f has a compatible system of p-power roots in S. We shall show that S_{perfd} solves the problem, i.e., that S_{perfd} is p-completely faithfully flat over R (and hence concentrated in degree 0) and is a perfectoid ring. In fact, by Remark 2.5 (2), it is enough to show that S_{perfd} is p-completely faithfully flat over R (as that forces S_{perfd}, and hence also $\Delta_{S/A,\mathrm{perf}}$, to be discrete).

It remains to show that S_{perfd} is p-completely faithfully flat over R. As $S_{\mathrm{perfd}} \simeq \Delta_{S/A,\mathrm{perf}} \otimes^L_A A/I$, it is enough to show that $A \to \Delta_{S/A,\mathrm{perf}}$ is (p, I)-completely faithfully flat. By stability of (p, I)-complete faithful flatness under direct limits and Frobenius twists (as A is perfect), it suffices to show that $\Delta_{S/A}$ is (p, I)-completely faithfully flat over A. By devissage, it even suffices to show that $\overline{\Delta}_{S/A}$ is p-completely faithfully flat over R. Using the Hodge-Tate filtration (Proposition VII.4.2), it is enough to show that each $\wedge^i L_{S/R}[-i]$ is p-completely faithfully flat over R. But $R \to S$ factors as $R \to R[x^{1/p^\infty}] =: R' \to S$ and $L_{R'/R}$ vanishes after derived p-completion (Exercise VII.2.2). By the transitivity triangle, we are reduced to showing that each $\wedge^i L_{S/R'}[-i]$ is p-completely faithfully flat over R. As S is itself p-completely faithfully flat over R, it suffices to show that $L_{S/R'}[-1]$ is simply isomorphic to S: the rest then follows as $\wedge^i(S[1])[-i] \simeq \Gamma^i_S(S)$ is also S-flat for all i. It remains to show that $L_{S/R'} \simeq S[1]$, but this follows as S is the quotient of R' by nonzerodivisor. □

Corollary 3.2 (Zariski closed sets are strongly Zariski closed)**.** *Let R be a perfectoid ring, and let $S = R/J$ be a p-complete quotient. Then there is a universal map $S \to S'$ with S' being a perfectoid ring. Moreover, this map is surjective.*

The key assertion here is the surjectivity of $S \to S'$; this implies in particular that the warning in [6, Remark II.2.4] is not necessary. The kernel of $S \to S'$ seems difficult to describe explicitly.

Proof. Work over the perfect prism (A, I) corresponding to R. We shall show that $S' := S_{\mathrm{perfd}}$ solves the problem. By Example 2.3 (2), it is enough to show that S_{perfd} is a perfectoid ring, and that $S \to S_{\mathrm{perfd}}$ is surjective.

To check that S_{perfd} is a perfectoid ring, using the observations in Remark 2.5 (1) and (2), we are reduced to checking that $\Delta_{S/A,\mathrm{perf}} \in D^{\leq 0}$. For this, it is enough to prove the stronger statement that $\Delta_{S/A} \in D^{\leq 0}$. By I-completeness, it suffices to show that $\overline{\Delta}_{S/A} \in D^{\leq 0}$. Thanks to the Hodge-Tate filtration and the stability of $D^{\leq 0}(R) \subset D(R)$ under p-completion, this reduces to showing that $\wedge^i L_{S/R}[-i] \in D^{\leq 0}$ for all i. But $R \to S$ is surjective, so $L_{S/R}[-1] \in D^{\leq 0}$ as $\Omega^1_{S/R} \simeq 0$; the rest follows formally as $D^{\leq 0}(R) \subset D(R)$ is stable under derived exterior powers.

We have shown that $S \to S_{\mathrm{perfd}}$ is the universal map from S to a perfectoid ring. It remains to check that $S \to S_{\mathrm{perfd}}$ is surjective. Assume first that the kernel $J \subset R$ of $R \to S$ is the p-completion of an ideal generated by a set $\{x_s^\sharp\}_{s\in S}$ of elements that lie in the image of the map $\sharp : R^\flat \to R$ (Definition IV.2.8). In this case, if J_∞ denotes the p-completion of the ideal generated by $\{x_s^{1/p^n,\sharp}\}_{s\in S,n\geq 0}$, then one checks directly that the R/J_∞ is perfectoid, and $S \simeq R/J \to R/J_\infty$ is the universal map from S to a perfectoid ring; this proves the assertion in this case. In general, as the surjectivity of a map of p-complete R-modules can be detected after p-completely faithfully flat base change, one uses Theorem 3.1 and Lemma 2.7 to reduce to the previously treated special case. □

References

[1] Y. André, *La conjecture de facteur direct*
[2] B. Bhatt, P. Scholze, *Projectivitiy of the Witt vector affine Grassmannian*
[3] B. Bhatt, *On the direct summand conjecture and its derived variant*
[4] J. Lurie, *Rational and p-adic homotopy theory*
[5] P. Scholze, *Perfectoid spaces*
[6] P. Scholze, *On torsion in the cohomology of locally symmetric spaces*