LECTURE VII: EXTENSION TO THE SINGULAR CASE VIA DERIVED PRISMATIC COHOMOLOGY

Let (A, I) be a bounded prism. In this lecture, we explain how to use Quillen's formalism of non-abelian derived functors [1, 11, 10] to extend the notion of prismatic cohomology to arbitrary (i.e., not necessarily smooth) p-complete A/I-algebras R in a manner compatible with the Hodge-Tate comparison theorem. In the following lectures, we shall use this extension to formulate and prove the étale comparison theorem for the prismatic cohomology of any p-complete A/I-algebra.

1. Non-abelian derived functors

Let A be a commutative ring. Write CAlg_A for the category of commutative A-algebras, and $\mathrm{Poly}_A \subset \mathrm{CAlg}_A$ for the full subcategory spanned by polynomial A-algebras in finitely many variables. Note that each such A-algebra P is a *projective object* in the category CAlg_A, i.e., for any surjection $B \to C$ in CAlg_A and any map $P \to C$ in CAlg_A, there exists a lift $P \to B$. In analogy with homological algebra, it is therefore natural to "derive" functors defined on Poly_A to functors defined on all of CAlg_A by simply applying the original functor to a suitable resolution. We discuss this construction (in limited generality) next using the language of ∞-categories; we refer the reader to Lurie's [8, §5.5.8] for an exhaustive modern treatement (see also [5, §9.2], [7, §2.1] for related expositions specialized to some examples also encountered below), Quillen's [11] for the original presentation via model categories.

Construction 1.1 (Non-abelian derived functors)**.** Consider a functor $F : \mathrm{Poly}_A \to \mathrm{Ab}$ on Poly_A valued in the category Ab of abelian groups. Our goal is to explain why F admits a well-behaved extension to all of CAlg_A. To formulate the result economically, we use the language of ∞-categories. Thus, let $\mathcal{D}(\mathrm{Ab})$ be the derived ∞-category of abelian groups. Note that $\mathcal{D}(\mathrm{Ab})$ admits all limits and colimits (and, in particular, has functorial cones). The desired extension of F is then captured by the following result, applicable more generally to functors valued in $\mathcal{D}(\mathrm{Ab})$.

Proposition 1.2. *Let $\mathcal{F} : \mathrm{Poly}_A \to \mathcal{D}(\mathrm{Ab})$ be a functor. There exists a unique extension $L\mathcal{F} : \mathrm{CAlg}_A \to \mathcal{D}(\mathrm{Ab})$ of the functor $L\mathcal{F}$ characterized by the following properties:*

(1) $L\mathcal{F}$ commutes with filtered colimits. In particular, if $A[S]$ is a polynomial algebra on a possibly infinite set S, then the canonical map $\mathrm{colim}_{\Sigma \subset S} \mathcal{F}(A[\Sigma]) \to L\mathcal{F}(A[S])$ is an equivalence, where the colimit ranges over all finite subsets $\Sigma \subset S$.

(2) $L\mathcal{F}$ commutes with geometric realizations of simplicial resolutions, i.e., given $B \in \mathrm{CAlg}_A$ with a simplicial resolution $P_\bullet \to B$ by A-algebras, the geometric realization[1] $|L\mathcal{F}(P_\bullet)|$ of $P_\bullet$ is equivalent to $L\mathcal{F}(B)$ via the natural map.

More categorically, the functor $L\mathcal{F} : \mathrm{CAlg}_A \to \mathcal{D}(\mathrm{Ab})$ is the left Kan extension of $\mathcal{F} : \mathrm{Poly}_A \to \mathcal{D}(\mathrm{Ab})$ along the inclusion $\mathrm{Poly}_A \subset \mathrm{CAlg}_A$. We call $L\mathcal{F}$ the left derived functor *of $\mathcal{F}$.*

[1]Given a simplicial object $X_\bullet : \Delta^{op} \to \mathcal{C}$ in an ∞-category $\mathcal{C}$, its geometric realization $|X_\bullet|$ is simply the colimit of $X_\bullet$. In our applications, we shall only apply this construction to a simplicial chain complex $K_\bullet \in \mathrm{Ch}(\mathrm{Ab})$, viewed as a simplicial object in $\mathcal{D}(\mathrm{Ab})$. In this case, there is a much more explicit and classical description of the colimit: the geometric realization $|K_\bullet|$ is simply the direct *sum* totalization of the bicomplex obtained from $K_\bullet$ by making the simplicial direction the horizontal one. Note that this bicomplex is located in the 2nd and 3rd quadrants with standard cohomological conventions. In particular, there are potentially infinitely many terms along antidiagonals, so using the direct *product* totalization would give a different notion.

Remark 1.3 (Explicitly computing $L\mathcal{F}$)**.** In applications, the functor $\mathcal{F}$ is strict, i.e., it is obtained from a functor $\mathcal{G} : \mathrm{Poly}_A \to \mathrm{Ch}(\mathrm{Ab})$ by passage to the derived category along the canonical functor $\mathrm{Ch}(\mathrm{Ab}) \to \mathcal{D}(\mathrm{Ab})$. For such $\mathcal{F}$, Proposition 1.2 gives a completely explicit recipe for computing $L\mathcal{F}(B)$ that can be phrased without ∞-categories as follows. For any $B \in \mathrm{CAlg}_A$, there is a functorial simplicial resolution $P_\bullet \to B$ by (typically infinitely generated) polynomial A-algebras P_i determined by $P_0 = A[B]$, $P_1 = A[A[B]]$, etc. with natural transition maps. Using Proposition 1.2 (1) and (2), we obtain the following concrete formula for $L\mathcal{F}(B)$: it is computed by the geometric realization of the simplicial chain complex $\mathcal{G}(P_\bullet)$, where $\mathcal{G}(P_n)$ is defined as $\mathrm{colim}_i\, \mathcal{G}(P_{n,i})$, where $\{P_{n,i}\}$ is the filtered collection of finitely generated polynomial subalgebras of P_n.

For future use, note that if $\mathcal{F}$ takes values in $\mathcal{D}^{\leq 0}(\mathrm{Ab})$, then the same holds true for $L\mathcal{F}$: this follows from Proposition 1.2 (1) and (2) as $\mathcal{D}^{\leq 0}(\mathrm{Ab}) \subset \mathcal{D}(\mathrm{Ab})$ is closed under all colimits.

Remark 1.4 (Variants)**.** Proposition 1.2 uses only basic category theory, and consequently has many variants. We discuss some relevant ones next.

(1) (Replacing the target) The target ∞-category $\mathcal{D}(\mathrm{Ab})$ can be replaced by any ∞-category $\mathcal{C}$ admitting all colimits. In applications, we shall often work with $\mathcal{C}$ being the category of derived I-complete objects in the derived ∞-category $\mathcal{D}(B)$ of B-modules, where B is a commutative ring and $I \subset B$ is a finitely generated ideal. Note that in this case, the colimits appearing in Proposition 1.2 must be interpreted in the completed sense, i.e., they are computed by applying derived I-completion to the corresponding colimit in $\mathcal{D}(B)$.

(2) (Enlarging the source) Proposition 1.2 also holds true if we replace CAlg_A with the ∞-category sCAlg_A of simplicial commutative A-algebras. In fact, in this case, the construction $\mathcal{F} \mapsto L\mathcal{F}$ establishes an equivalence between $\mathrm{Fun}(\mathrm{Poly}_A, \mathcal{C})$ and the full ∞-subcategory of $\mathrm{Fun}(\mathrm{sCAlg}_A, \mathcal{C})$ spanned by functors commuting with sifted colimits. To avoid additional technical baggage, we shall largely avoid simplicial commutative rings in these lectures. However, the author finds it useful to work in the larger category sCAlg_A when deriving functors as colimits (such as "reducing modulo p") behave better in this setting.

(3) (Derived exterior powers) We may replace the inclusion $\mathrm{Poly}_A \subset \mathrm{CAlg}_A$ with other inclusions. For example, there is an analog of Proposition 1.2 for the inclusion $\mathrm{Vect}_A \subset \mathrm{Mod}_A$ of the category Vect_A of finite projective A-modules inside all A-modules. If the functor $\mathcal{F}$ is additive and right exact, then $L\mathcal{F}$ gives the usual derived functor from homological algebra. On the other hand, as there is no additivity assumption in Proposition 1.2, we are now allowed to derive non-additive functors. An example that will be relevant later is given by $M \mapsto \wedge^i M$ for any $i \geq 0$; the corresponding derived functor $L\wedge^i : \mathrm{Mod}_A \to \mathcal{D}(A)$ give a notion of derived exterior power for A-modules that agrees with the usual notion for flat A-modules (but not in general). In fact, as in (2) above, we may even extend this to derived category, getting functors $L\wedge^i : \mathcal{D}^{\leq 0}(A) \to \mathcal{D}(A)$ for $i \geq 0$; in this derived setting, we simply write $\wedge^i$ instead of $L\wedge^i$ if there is no potential for confusion. There are also variants for other tensorial constructions, such as symmetric powers Sym^i or divided powers Γ^i; see [10, §I.V.4] for more on these notions and the interrelationship between them.

2. The cotangent complex

In this section, we discuss the first example where Construction 1.1 arose in algebraic geometry.

Definition 2.1 (The cotangent complex)**.** Let A be a commutative ring. The *cotangent complex* functor $L_{-/A} : \mathrm{CAlg}_A \to \mathcal{D}(A)$ is the left derived functor of the functor $\mathrm{Poly}_A \to \mathrm{Mod}_A \subset \mathcal{D}(A)$ given by $B \mapsto \Omega^1_{B/A}$.

Using the explicit description of left derived functor given following Proposition 1.2, it is not too difficult to see that $L_{B/A}$ is naturally an object of $D^{\leq 0}(B)$ (and not just $D^{\leq 0}(A)$). We summarize the fundamental properties of this construction next, and refer to [10, 12] for proofs.

(1) (Recovering Kähler differentials). We have $H^0(L_{B/A}) \simeq \Omega^1_{B/A}$.

(2) (Acyclicity of smooth algebras). If $A \to B$ is smooth, then $L_{B/A} \simeq \Omega^1_{B/A}[0]$. Thus, smooth algebras are "acyclic" for $L_{-/A}$. In particular, if $A \to B$ is étale, then $L_{B/A} \simeq 0$.

(3) (Transitivity triangle). Given $A \to B \to C$, we have an exact triangle

$$L_{B/A} \otimes^L_B C \to L_{C/A} \to L_{C/B}$$

in $D(C)$, extending the standard low degree exact sequences for Kähler differentials.

(4) (Behaviour on quotients). If $A \to B$ is surjective, then $H^{-1}(L_{B/A}) \simeq I/I^2$, where $I \subset A$ is the kernel. If I is generated by a regular sequence, then $H^i(L_{B/A}) \simeq 0$ for all $i \neq -1$; in particular, $L_{B/A}[-1]$ is a finite projective B-module.

(5) (Base change). Given maps $A \to B$ and $A \to C$ that are Tor independent, we have a natural base change isomorphism

$$L_{C/A} \otimes^L_A B \simeq L_{C \otimes_A B/B}.$$

In particular, this applies if one of the map $A \to B$ or $A \to C$ is flat. (More generally, a similar assertion holds true without any Tor independence conditions provided one uses the simplicial commutative ring $C \otimes^L_A B$ in lieu of $C \otimes_A B$.)

Although this plays no role in our lectures, we remark for the sake of completeness that the cotangent complex has a natural interpretation in the world of derived algebraic geometry: it classifies derivations in the derived sense, see [9, Chapter 17].

Exercise 2.2. Show that if $A \to B$ is a map of perfect $\mathbf{F}_p$-algebras, then $L_{B/A} \simeq 0$. Using the trick used in the proof of Proposition IV.2.11, deduce that if $C \to D$ is a map of perfectoid rings, then $L_{D/C}$ vanishes after derived p-completion. (See [6, Lemma 3.14] for a proof.)

Remark 2.3 (Derived i-forms)**.** For any integer $i \geq 0$, one can also derive the functor $B \mapsto \Omega^i_{B/A}$ of i-forms (with Definition 2.1 corresponding to $i = 1$). The corresponding derived functor is denoted $\wedge^i L_{-/A}$. One can show that $\wedge^i L_{B/A}$ coincides with the notion defined in Remark 1.4 (3), so the notation is not ambiguous.

Remark 2.4 (Compatibility with p-adic completion)**.** Let A be a p-adically complete commutative ring with bounded p^∞-torsion. Let B be a flat A-algebra, so B also has bounded p^∞-torsion. Let $\widehat{B}$ be the p-adic completion of B. Then the cotangent complex of the map $B \to \widehat{B}$ vanishes after derived p-completion: this is true after applying $- \otimes^L_{\mathbf{Z}} \mathbf{Z}/p$ via the base change formula for the cotangent complex, and the rest follows by derived Nakayama. In particular, if B is a smooth A-algebra, then the derived p-completion of $L_{\widehat{B}/A}$ is a finite projective $\widehat{B}$-module.

3. Derived de Rham cohomology

Fix a base ring k of characteristic p. The following definition (as well as the Hodge complete variant in Remark 3.8 below) was first studied in [10].

Definition 3.1 (Derived de Rham cohomology)**.** The *derived de Rham cohomology* functor $\mathrm{dR}_{-/k}$: $\mathrm{CAlg}_k \to \mathcal{D}(k)$ is the left derived functor of the functor $\mathrm{Poly}_k \to \mathcal{D}(k)$ given by $R \mapsto \Omega^*_{R/k}$.

Unwinding definitions, this is computed as follows: for any k-algebra R, if $P_\bullet \to R$ denotes the canonical simplicial resolution of R by polynomial k-algebras, then $\mathrm{dR}_{R/k}$ is computed by the direct sum totalization of the second quadrant bicomplex (attached to) $\Omega^*_{P_\bullet/k}$. Using the $P_n^{(1)}$-linearity of each $\Omega^*_{P_n/k}$ (see Construction VI.1.9), we may regard $\mathrm{dR}_{R/k}$ as an $R^{(1)}$-linear complex, where $R^{(1)} := |P_\bullet^{(1)}| \simeq R \otimes^L_{k,\phi} k$ is the derived Frobenius twist of R. The basic result about this object is that the Cartier isomorphism derives well [3].

Proposition 3.2 (Derived Cartier isomorphism)**.** *For any $R \in \mathrm{CAlg}_k$, there is a functorial increasing multiplicative exhaustive filtration Fil^{conj}_* on $\mathrm{dR}_{R/k}$ in $\mathcal{D}(R^{(1)})$ equipped with canonical identifications $\mathrm{gr}^i_{conj}\mathrm{dR}_{R/k} \simeq \wedge^i L_{R^{(1)}/k}[-i]$.*

The filtration from Proposition 3.2 is called the *conjugate filtration.* Before constructing it, let us briefly explain how one (formulates and) justifies assertions concerning filtrations on objects in the derived category.

Construction 3.3 (Filtered derived category)**.** Write $\mathcal{C} := \mathrm{Fun}(\mathbf{N}, \mathcal{D}(k))$ for the ∞-category of diagrams $\{F_n\} := \{F_0 \to F_1 \to F_2 \to ...\}$ in $\mathcal{D}(k)$; this is a presentable stable ∞-category (and thus its homotopy category is a triangulated category sometimes called the filtered derived category[2]). The construction carrying $\{F_n\} \in \mathcal{C}$ to $F_\infty := \mathrm{colim}_n F_n$ gives an colimit preserving functor $\mathcal{C} \to \mathcal{D}(k)$ that we call the "underlying object" functor. Likewise, if we set $\mathrm{gr}_i(\{F_n\}) := \mathrm{cone}(F_{i-1} \to F_i)$, then extracting any individual F_i or $\mathrm{gr}_i(\{F_n\})$ from $\{F_n\}$ gives colimit preserving functors $\mathrm{ev}_i, \mathrm{gr}_i : \mathcal{C} \to \mathcal{D}(k)$. A basic fact about this construction is the following: a map $f : \{F_n\} \to \{G_n\}$ in $\mathcal{C}$ is an isomorphism if and only if $\mathrm{gr}_i(f)$ is so for all i.

In this language, Proposition 3.2 says that the derived de Rham complex functor $\mathrm{CAlg}_k \to \mathcal{D}(k)$ has a preferred lift $\{\mathrm{Fil}^{conj}_n\}$ along the "underlying object" functor $\mathcal{C} \to \mathcal{D}(k)$ with graded pieces as described in the proposition. Let us explain how to construct this using the Cartier isomorphism for polynomial k-algebras.

Sketch of proof of Proposition 3.2. For any $R \in \mathrm{Poly}_k$, equipping the de Rham complex $\Omega^*_{R/k}$ with its canonical filtration (also called the conjugate filtration) naturally defines an object $\{F_n(R)\} \in \mathcal{C}$. Explicitly, we have $\mathrm{ev}_i(\{F_n(R)\}) = \tau^{\leq i}\Omega^*_{R/k}$ with obvious transition maps. Via the Cartier isomorphism (Construction VI.1.9), we have a natural isomorphism $\mathrm{gr}_i(\{F_n\}) \simeq \Omega^i_{R^{(1)}/k}[-i]$ for all $i \geq 0$. This construction can be regarded as a lift $\mathrm{Poly}_k \to \mathcal{C}$ of the de Rham complex functor $\Omega^*_{-/k} : \mathrm{Poly}_k \to \mathcal{D}(k)$ under the "underlying object" functor $\mathcal{C} \to \mathcal{D}(k)$ of passing to the underlying object. As the various functors relating $\mathcal{C}$ with $\mathcal{D}(k)$ introduced in Construction 3.3 are colimit preserving, passing to left derived functors now gives the proposition. ☐

As the Cartier isomorphism actually holds true for all smooth k-algebras, it follows that derived de Rham complex coincides with the usual de Rham complex in the smooth case:

Corollary 3.4. *If R is a smooth k-algebra, then there is a canonical isomorphism $\mathrm{dR}_{R/k} \simeq \Omega^*_{R/k}$ identifying the conjugate filtration on either side.*

Proof. By construction (or the universal property of left Kan extensions), we have a natural (in R) map $\mathrm{dR}_{R/k} \to \Omega^*_{R/k}$ in $\mathcal{D}(k)$. In fact, as in the proof of Proposition 3.2, this map even lifts to the filtered derived category if one equips either side with the conjugate filtration. On the graded pieces, this map is given by the natural map $\wedge^i L_{R^{(1)}/k}[-i] \to \Omega^i_{R^{(1)}/k}[-i]$; here we use the Cartier isomorphism for the smooth k-algebra R to identify the right hand side. As R is smooth over k, this last map is an isomorphism for all i, and thus $\mathrm{dR}_{R/k} \simeq \Omega^*_{R/k}$ as conjugate filtered objects. ☐

[2]This triangulated category can be realized as the derived category of the abelian category $\mathrm{Fun}(\mathbf{N}, \mathrm{Mod}_k)$.

One reason to contemplate derived de Rham cohomology is that there exists a large supply of rings whose derived de Rham cohomology is concentrated in degree 0, and can thus be described explicitly. The most important example is the following:

Example 3.5 (Regular semiperfect rings)**.** Let k be a perfect ring. Let S be a k-algebra of the form R/I where R is a perfect k-algebra and $I \subset R$ is an ideal generated by a regular sequence; we call such rings *regular semiperfect*. Then $L_{S/k} \simeq L_{S/R}$ by the transitivity triangle since $L_{R/k} \simeq 0$ (Lemma II.3.5). As I is generated by a regular sequence, we have $L_{S/R} \simeq I/I^2[1]$, with I/I^2 being a finite projective S-module. A standard lemma with derived exterior powers then shows that $\wedge^i L_{S/R} \simeq \Gamma^i_R(I/I^2)[i]$. In particular, for any $i \geq 0$, the complex $\wedge^i L_{S/k}[-i]$ has homology only in degree 0. The conjugate filtration on $\mathrm{dR}_{S/k}$ then shows that $\mathrm{dR}_{S/k}$ also has homology only in degree 0, and can thus be identified with a commutative k-algebra. In fact, it is possible to calculate this ring explicitly [3, 7]: it coincides with $D_I(R)$, the divided power envelope of $I \subset R$ (see Construction VI.1.1 for an integral variant). For example, if $S = k[x^{1/p^\infty}]/(x)$, then

$$\mathrm{dR}_{S/k} \simeq D_{(x)}(k[x^{1/p^\infty}]) \simeq \bigoplus_{i \in \mathbf{N}[1/p]} k \cdot \frac{x^i}{\lfloor i \rfloor !}.$$

More generally, a similar description can be given for any S as above.

The utility of the previous example is that we can cover smooth algebras by regular semiperfect rings, and then calculate derived de Rham cohomology in the smooth case by descent.

Remark 3.6 (Computing derived de Rham cohomology by descent)**.** Assume k is a perfect ring. Let R be a smooth k-algebra with perfection $S := R_{\mathrm{perf}}$. Let $S^\bullet$ be the Cech nerve of $R \to S$. Since R is smooth over k, the map $R \to S$ is flat (easy direction of Kunz's theorem) and each S^n is regular semiperfect in the sense of Example 3.5. (For example, if $R = k[x]$, then $S^1 := k[u,v]_{perf}/(u-v)$.) In particular, each $\mathrm{dR}_{S^n/k}$ is concentrated in degree 0 by Example 3.5. One can also show [7, §3] that the functor $\mathrm{dR}_{-/k}$ satisfies descent with respect to this cover. Combining this with Example 3.5 shows that $\mathrm{dR}_{R/k}$ can be computed by the complex $\mathrm{dR}_{S^\bullet/k}$. Of course, we also know that $\mathrm{dR}_{R/k}$ can be calculated by the complex $\Omega^*_{R/k}$ (Corollary 3.4), which is a much "smaller" complex. The advantage of the description via $S^\bullet$ is that it passes to crystalline cohomology (see Remark 3.7).

Remark 3.7 (Derived crystalline cohomology)**.** Assume k is a perfect ring. Fix a p-torsionfree ring A lifting k, i.e., $A/p \simeq k$. In §VI.1, we explained why the de Rham complex of smooth k-algebras has a preferred lift to (the derived category of) A given by the theory $R\Gamma_{crys}(-/A)$ of crystalline cohomology. As in Definition 3.1, one can derive this functor[3] to define derived crystalline cohomology $R\Gamma_{dcrys}(-/A)$ for any k-algebra A. The de Rham crystalline comparison for smooth k-algebras (Corollary VI.1.8) then derives to an identification $R\Gamma_{dcrys}(-/A) \otimes^L_A k \simeq \mathrm{dR}_{-/k}$ of functors, leading to analogs of Corollary 3.4, Example 3.5 and Remark 3.6 above. In particular,for any smooth k-algebra R, there is a canonical and explicit complex computing $R\Gamma_{crys}(R/k)$: we simply apply $R\Gamma_{dcrys}(-/k)$ to the Cech nerve $S^\bullet$ of $R \to S := R_{perf}$. In fact, one can also give an explicit formula for $R\Gamma_{dcrys}(S/A)$ when S is regular semiperfect, though we do not do so here. Note that calculating $R\Gamma_{crys}(R/A)$ via the complex $R\Gamma_{dcrys}(S^\bullet/A)$ is slightly better than Construction VI.1.4: the latter entailed making the auxiliary choice of the surjection $P \to R$ from a free A-algebra. (Even though P could be chosen functorially in A leading to a strictly functorial theory as discussed in Lecture VI, there is no preferred choice of such a functor.)

[3] To avoid any confusion, we emphasize that we are still working with characteristic p input rings, i.e., we derive the functor $R\Gamma_{crys}(-/A)$ along the inclusion $\mathrm{Poly}_k \subset \mathrm{CAlg}_k$.

Remark 3.8 (The characteristic 0 picture)**.** The discussion above crucially relies on the Cartier isomorphism. In particular, even though Definition 3.1 makes sense over any base ring k, the conclusion of Corollary 3.4 is completely false in characteristic 0. In fact, if k has characteristic 0, then $k \simeq \Omega^*_{A/k}$ for any polynomial k-algebra A by the acyclicity of affine space, which formally implies that $k \simeq \mathrm{dR}_{A/k}$ for all k-algebras A in this setting. This behaviour arises because Definition 3.1 is adapted to the conjugate filtration, which behaves well only in characteristic p. To fix this, in characteristic 0, one must force the Hodge filtration to converge. Even though this story will play no role in these lectures, we give a quick summary next; it first appeared in p-adic Hodge theory in Beilinson's proof [2] of Fontaine's C_{dR}-conjecture.

Let k be a ring of characteristic 0. For any polynomial k-algebra A, the stupid filtration Fil^*_H on $\Omega^*_{A/k}$ gives a descending $\mathbf{N}$-indexed filtration on $\Omega^*_{A/k}$ with gr^i_H given by $\Omega^i_{A/k}[-i]$. Note that this filtration is complete, i.e., $\lim_i \mathrm{Fil}^i_H \simeq 0$. The association $A \mapsto \Omega^*_{A/k}$ gives a functor on polynomial k-algebras A as valued in $\widehat{DF}(k)$, the ∞-category of objects in $\mathcal{D}(k)$ equipped with a complete descending $\mathbf{N}$-indexed filtration (defined by a variant of Construction 3.3, see [7, §5.1]). Taking the left derived functor then allows us to define the *Hodge completed derived de Rham complex* $\widehat{\mathrm{dR}}_{A/k} \in \mathcal{D}(k)$; by construction, this object comes equipped with a complete descending $\mathbf{N}$-indexed filtration whose graded pieces are given by $\wedge^i L_{A/k}[-i]$. Unwinding definitions, this is computed as follows: for any k-algebra A, if $P_\bullet \to A$ denotes the canonical simplicial resolution of A by polynomial k-algebras, then $\widehat{\mathrm{dR}}_{A/k}$ is computed by the direct product totalization of the second quadrant bicomplex (attached to) $\Omega^*_{P_\bullet/k}$. One can show [4] that if k is a field and A is a finitely generated k-algebra, then $\widehat{\mathrm{dR}}_{A/k}$ computes the "correct" cohomology theory for $\mathrm{Spec}(A)$ (for example, it coincides with Betti cohomology when $k = \mathbf{C}$).

4. Derived prismatic cohomology

Let (A, I) be a bounded prism. In the previous lectures, we have constructed the prismatic cohomology functor $R \mapsto \Delta_{R/A}$ defined on formally smooth A/I-algebras R and taking values in the category $D_{comp}(A)$ of (p, I)-complete commutative algebra objects in $D(A)$ (and similarly for the Hodge-Tate variant). We now extend this notion to the singular case[4].

Definition 4.1 (Derived prismatic cohomology)**.** The *derived prismatic cohomology* functor $L\Delta_{-/A} : \mathrm{CAlg}_{A/I} \to \mathcal{D}_{comp}(A)$ is the left derived functor of the functor $\mathrm{Poly}_{A/I} \to \mathcal{D}_{comp}(A)$ given by $R_0 \mapsto \Delta_{\widehat{R_0}/A}$, where $\widehat{R_0}$ is the p-adic completion of R_0. The *derived Hodge-Tate cohomology* functor $L\overline{\Delta}_{-/A} : \mathrm{CAlg}_{A/I} \to \mathcal{D}_{comp}(A/I)$ is defined similarly.

Let us note some easy properties that can be derived from the smooth case. First, $L\Delta_{R/A}$ and $L\overline{\Delta}_{R/A}$ are naturally commutative algebras in $\mathcal{D}_{comp}(A)$ and $\mathcal{D}_{comp}(R)$ respectively. Moreover, we have $L\Delta_{R/A} \otimes^L_A A/I \simeq L\overline{\Delta}_{R/A}$ via the natural map. Finally, $L\Delta_{R/A}$ comes equipped with a ϕ_A-semilinear "Frobenius" endomorphism ϕ_R. (In fact, one can show that ϕ_R lifts to a structure making $L\Delta_{R/A}$ into a "derived δ-ring", but we do not make this precise here.) The main tool for controlling the behaviour of this construction is the following:

Proposition 4.2 (The Hodge-Tate comparison)**.** *For any $R \in \mathrm{CAlg}_{A/I}$, the complex $L\overline{\Delta}_{R/A}$ comes equipped with a functorial increasing multiplicative exhaustive filtration Fil^{HT}_* in $\mathcal{D}_{comp}(R)$ and canonical identifications $\mathrm{gr}^{HT}_i(L\overline{\Delta}_{R/A}) \simeq \wedge^i L_{R/(A/I)}\{-i\}[-i]^\wedge$ (where the twists $\{-i\}$ were those introduced in Remark V.3.7).*

[4]To avoid discussing variants of Proposition 1.2 involving categories of p-adically complete rings, we simply define our desired extension on all commutative A/I-algebras using p-adic completions. A better approach would entail contemplating the left Kan extension of $\Delta_{-/A}$ from formally smooth A/I-algebras to p-complete A/I-algebras.

Proof. This follows just as Proposition 3.2 once we know the following: if R is the p-adic completion of a polynomial A-algebra, we have $L\overline{\Delta}_{R/A} \simeq \overline{\Delta}_{R/A}$ via the canonical map. We leave this as an exercise to the reader in unwinding definitions and using Remark 2.4 (as well as variants for higher exterior powers). ☐

As a result, we have not changed the theory in the smooth case.

Corollary 4.3. *If R is a formally smooth A/I-algebra, then $L\Delta_{R/A} \simeq \Delta_{R/A}$ compatibly with all the structure.*

Proof. This follows just like Corollary 3.4 using Proposition 4.2 instead of Proposition 3.2. ☐

Thanks to the corollary, we are allowed to abuse notation: from now, we simply write $\Delta_{R/A}$ for $L\Delta_{R/A}$ as there is no potential for confusion (and similarly for the Hodge-Tate variant $\overline{\Delta}_{S/A}$). As in the case of derived de Rham cohomology, one advantage of derived prismatic cohomology is the existence of a large supply of rings where the theory is concentrated in degree 0.

Example 4.4 (Regular semiperfectoid rings)**.** Assume $(A, (d))$ is a perfect prism. Let S be an $A/(d)$-algebra of the form R/J, where R be a perfectoid $A/(d)$-algebra and $J \subset R$ is an ideal generated by a regular sequence; we shall call such rings *regular semiperfectoid*. Assume for simplicity that $A/(d)$ is p-torsionfree and that S is p-completely flat over $A/(d)$. The Hodge-Tate comparison shows that $\overline{\Delta}_{S/A}$ admits an increasing exhaustive filtration with graded pieces $\wedge^i L_{S/A}\{-i\}[-i]$. By the assumption on S, each of these pieces is a finite projective S-module as in Example 3.5. It follows that $\overline{\Delta}_{S/A}$ is concentrated in degree 0 and given by a p-completely flat S-algebra. Consequently, $\Delta_{S/A}$ is concentrated in degree 0 and given by a (p, d)-completely flat A-algebra. In fact, one can also show that the Frobenius ϕ_R on prismatic cohomology makes $\Delta_{S/A}$ into a δ-A-algebra, so $(\Delta_{S/A}, (d))$ is a flat prism over $(A, (d))$. It is an interesting problem to describe $\Delta_{S/A}$ explicitly as a ring (i.e., via generators and relations) in analogy with Example 3.5. While we do not have a good general answer to this, it is possible to relate $\Delta_{S/A}$ to constructions we have encountered earlier: one can show that $\Delta_{S/A} \simeq A_{\mathrm{inf}}(R)\{\frac{f_1}{d}, ..., \frac{f_r}{d}\}^\wedge$ (see Corollary VI.2.3), where $f_1, ..., f_r \in A_{\mathrm{inf}}(R)$ is the lift of a regular sequence in R generating I.

Remark 4.5 (Semiperfectoid rings)**.** Assume $(A, (d))$ is a perfect prism. Let S be a p-complete $A/(d)$-algebra of the form R/J with R perfectoid; we call such rings *semiperfectoid*. In this case, a variant of the argument used in Examples 3.5 and 4.4 shows that $\Delta_{S/A} \in D^{\leq 0}(A)$. If S is not regular semiperfectoid, then $\Delta_{S/A}$ is typically not concentrated in degree 0; in fact, it can often have cohomology in infinitely many negative degrees.

In a later lecture, we shall combine descent idea from Remark 3.7 with Remark 4.5 to prove the étale comparison theorem for prismatic cohomology of arbitrary p-complete A/I-algebras.

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