LECTURE VI: THE HODGE-TATE AND CRYSTALLINE COMPARISON THEOREMS

Last time, we formulated the following theorem (Theorem V.3.8).

Theorem 0.1 (The Hodge-Tate comparison theorem)**.** *Let $(A,(d))$ be a bounded prism, and let R be a formally smooth $A/(d)$-algebra. The Hodge-Tate comparison maps give an isomorphism*

$$\eta_R^* : (\Omega_R^*, d_{dR}) \to (H^*(\overline{\Delta}_{R/A}), \beta_d)$$

of complexes. In particular, $\overline{\Delta}_{R/A} \in D(R)$ is a perfect complex.

The goal of this lecture is to sketch some aspects of the proof of this result. We emphasize that these notes do not contain a complete proof of the result in the above generality. Instead, we have chosen to emphasize the key ideas, and suppress the (somewhat technical) commutative algebra arguments by imposing additional assumptions at various points.

1. A quick recollection of crystalline cohomology

In this section, we fix p-torsionfree ring A as well as a smooth A/p-algebra R. Our goal is to give a construction of crystalline cohomology of R relative to A. For reasons of time and space, we do not define the latter via the standard site-theoretic construction (as in [1]). Instead, we shall simply construct an explicit complex that computes the cohomology of the structure sheaf on the crystalline site $(R/A)_{crys}$. Towards this end, we shall need the notion of divided power envelopes; we recall this next in the limited generality sufficient for our purposes.

Construction 1.1 (Divided power envelopes)**.** Let P be an ind-smooth[1] A-algebra equipped with a surjection $P \to R$ with kernel $J \subset P$. Write $D_J(P)$ for the p-adic completion of the subring of $P[1/p]$ generated by P and the divided powers $\gamma_n(x) := \frac{x^n}{n!}$ for all $n \geq 1$ and $x \in J$. Note that $D_J(P)$ is p-torsionfree by construction. One checks that there is a natural map $D_J(P) \to R$ factoring $P \to R$ defined by sending $\gamma_n(x) \in D_J(P)$ to 0 for all $x \in J$ and $n \geq 1$, and that the kernel of $D_J(P) \to R$ itself has divided powers. Moreover, this construction is the *PD-envelope* of $P \to R$, i.e., it has the following universal property:

Lemma 1.2. *Let $D \to R$ be a surjection of A-algebras with p nilpotent on D. Assume we are given a divided power structure on the kernel I[2]. Then any A-algebra map $P \to D$ compatible with the maps to R extends uniquely to a map $D_J(P) \to D$ compatibly with the divided power structures.*

In particular, for R fixed, the construction carrying $(P \to R)$ to $(D_J(P) \to R)$ can be thought of as a functor on the category of all surjections $P \to R$ as above. For future reference, let us remark that there is a natural flat connection $D_J(P) \to D_J(P)\widehat{\otimes}_P \Omega^1_{P/A}$ on $D_J(P)$ extending the obvious one on P via the observation that $d(\gamma_n(x)) = \gamma_{n-1}(x)dx$ in $\Omega^1_{P/A}[1/p]$ for all $x \in J$ and $n \geq 1$.

[1]For computational purposes, smooth algebras (or even polynomial algebras in finitely many variables) suffice. However, to get a strictly functorial complex, it is convenient to allow polynomial algebras in infinitely many variables.

[2]This means we are given maps $\gamma_n : I \to I$ for $n \geq 1$ such that $\gamma_n(x) \in I$ mimics the properties of $\frac{x^n}{n!}$; see [7, Tag 07GL] for a list of required properties. We always demand our divided power structures to be compatible with those on (p), i.e., for $x \in (p)$, the divided power structure is the one determined by the observation that $\gamma_n(p) \in \mathbf{Z}_{(p)}$ for all $n \geq 1$ as $n \geq \mathrm{val}_p(n!)$. One consequence of the existence of divided powers on I coupled with the assumption $p^n = 0$ in D for some $n \geq 0$ is that each element of I is nilpotent: we have $x^{p^n} = (p^n)! \cdot \gamma_{p^n}(x) = 0$. It is not difficult to then show that $D \to R$ is an inductive limit of nilpotent thickenings $D_i \to R$ of R, where D_i ranges over all sufficiently large finitely generated A-subalgebras of D.

Example 1.3 (The divided power polynomial algebra)**.** Say $R = A/p$, and $P = A[x]$ with the map $P \to R$ defined by $x \mapsto 0$. Then $J = (x)$, and $D_J(P)$ is the p-adic completion of $A[\{\gamma_n(x)\}_{n\geq 1}] \subset A[1/p]$. Using the grading on P, this simplifies to $D_J(P) = \widehat{\bigoplus}_{n\geq 0} A \cdot \gamma_n(x)$, where the multiplication on the right hand side is the obvious one: $\gamma_n(x) \cdot \gamma_m(x) = \frac{(nm)!}{n!m!} \cdot \gamma_{nm}(x)$. Note that this ring is not noetherian: the kernel of $D_J(P) \to R$ is topologically generated by all the $\gamma_n(x)$'s, and is not finitely generated. Nevertheless, the main virtue of the divided power envelope construction is evident in this example: the Poincaré lemma holds true, i.e., the de Rham complex $D_J(P) \to D_J(P) \otimes_P \Omega^1_{P/A}$ has homology only in degree 0 given by the constants $A \cdot 1 \subset D_J(P)$. In contrast, if $D_J(P)$ were replaced by the power series ring $\lim_n P/J^n$, the de Rham complex would have infinite dimensional homology (as the cycles $x^{kp-1}dx$ are not boundaries for any $k \geq 1$).

Let us use this notion go construct crystalline cohomology via a Cech-Alexander complex.

Construction 1.4 (Computing crystalline cohomology)**.** Choose a surjection $P \to R$ with P being an ind-smooth A-algebra. Geometrically, such a map can be obtained by embeding Spec(R) into the affine space Spec(P) over Spec(A). Let $P^\bullet$ for the Cech nerve of $A \to P$, i.e.,

$$P^\bullet := \Big(P \rightrightarrows P \otimes_A P \substack{\rightarrow\\\rightarrow\\\rightarrow} P \otimes_A P \otimes_A P \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \Big),$$

viewed as a cosimplicial A-algebra. Note that each P^n is an ind-smooth A-algebra. For each $n \geq 0$, we then have induced surjections $P^n := P^{\otimes_A (n+1)} \xrightarrow{\mu} P \to R$, where μ is the multiplication map; these surjections are compatible with the structure maps of $P^\bullet$. Thus, if we write $J^n \subset P^n$ for the kernel of this map, then we obtain a cosimplicial ideal $J^\bullet \subset P^\bullet$ with $P^\bullet/J^\bullet$ being identified with R (viewed as a constant cosimplicial ring). Applying Construction 1.1 termwise, we obtain another cosimplicial A-algebra

$$C^\bullet_{crys}(R/A) := D_{J^\bullet}(P^\bullet) := \Big(D_{J^0}(P^0) \rightrightarrows D_{J^1}(P^1) \substack{\rightarrow\\\rightarrow\\\rightarrow} D_{J^2}(P^2) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \Big). \tag{1}$$

The crystalline cohomology of R is defined to be the cohomology of the associated complex. To distinguish between a cosimplicial abelian group and the associated chain complex, we write

$$R\Gamma_{crys}(R/A) := \mathrm{Tot}(C^\bullet_{crys}(R/A)) \in D(A).$$

By general properties of the Dold-Kan construction, this is a commutative algebra object of $D(A)$.

Remark 1.5. To assuage the readers worried about functoriality issues inherent in Construction 1.4, we note that the initial surjection $P \to R$ appearing above can be choosen functorially in R: we may simply take P to be the free A-algebra on the underlying set of A. With such a choice, the cosimplicial rings $P^\bullet$ and $D_{J^\bullet}(P^\bullet)$, and thus also the crystalline cohomology complex $R\Gamma_{crys}(R/A)$, are strictly functorial in the smooth A/p-algebra R. In particular, the Frobenius endomorphism of R induces a ϕ_A-semilinear endomorphism $R\Gamma_{crys}(R/A)$.

The basic theorem about crystalline cohomology is the following result; we refer to [1] for a full exposition of the relevant theory, and [2] for a somewhat quicker argument for the comparison using only Cech theory and the Poincaré lemma.

Theorem 1.6 (The de Rham comparison for crystalline cohomology)**.** *For any surjection $P \to R$ with P ind-smooth over A and kernel J, there is a natural quasi-isomorphism between $R\Gamma_{crys}(R/A)$ and the de Rham complex $\Omega^*_{P/A} \widehat{\otimes}_P D_J(P)$ (where the latter makes sense thanks to the flat connection on $D_J(P)$ mentioned in Construction 1.1).*

Remark 1.7. For the reader unfamiliar with the crystalline theory, we note that Theorem 1.6 has a variant in characteristic 0 that is somewhat easier to understand. In this variant, if R is a smooth k-algebra with k being a field of characteristic 0, one simply runs Construction 1.4 by taking $A = A/I = k$ and replacing the pd-envelope functor $(J, P) \mapsto D_J(P)$ with the completion functor $(J, P) \mapsto \widehat{P}_J := \lim_n P/J^n$. The analog of (1) then computes Grothendieck infinitesimal cohomology $R\Gamma_{inf}(R/k)$, and the analog of Theorem 1.6 holds true in this setting as the Poincaré lemma holds true for the power series rings in characteristic 0.

Applying Theorem 1.6 to P being a smooth lift of R to A, we learn how to compute crystalline cohomology modulo p.

Corollary 1.8 (The crystalline-de Rham comparison modulo p)**.** *There is a natural*[3] *quasi-isomorphism* $R\Gamma_{crys}(R/A) \otimes^L_A A/p \simeq \Omega^*_{R/(A/p)}$ *of commutative algebra objects in* $D(A/p)$.

Let us use Corollary 1.8 to formulate the statement of the Cartier isomorphism, which is perhaps the most useful tool for controlling de Rham cohomology in characteristic p; we refer to [5, §7] for a detailed exposition, and [4] for an application.

Construction 1.9 (The Cartier isomorphism)**.** Write $\phi_A : A/p \to A/p$ for the Frobenius, and let $R^{(1)} := R \otimes_{A,\phi} A$ be the Frobenius twist. There is a natural relative Frobenius $\phi : R^{(1)} \to R$, and the de Rham complex $\Omega^*_{R/A}$ can be viewed as an $R^{(1)}$-linear complex $d(\phi(x)) = 0$ for all $x \in R^{(1)}$. In particular, the strictly[4] graded commutative ring $H^*(\Omega^*_{R/(A/p)})$ is naturally an $R^{(1)}$-algebra. Moreover, via Corollary 1.8 and similarly to Construction V.3.5, we have an induced A/p-linear Bockstein differential β_p on $H^*(\Omega^*_{R/(A/p)})$, making the latter into a strictly graded commutative A/p-dga. The universal property of the de Rham complex then gives a map

$$\mathrm{Cart}^* : (\Omega^*_{R^{(1)}/(A/p)}, d_{dR}) \to (H^*(\Omega^*_{R/(A/p)}), \beta_p) \tag{2}$$

of A/p-dgas. Cartier's theorem is that the above map[5] is an isomorphism. In particular, the i-th de Rham cohomology group $H^i(\Omega^*_{R/(A/p)})$ is canonically identified with the module $\Omega^i_{R^{(1)}/(A/p)}$ of i-forms on $R^{(1)}$.

Remark 1.10. Note that one can also formulate the Cartier isomorphism from Construction 1.9 as an isomorphism

$$\Omega^*_{R^{(1)}/(A/p)} \simeq H^*(R\Gamma_{crys}(R/A) \otimes^L_A A/p)$$

of graded rings. Either formulation of the Cartier isomorphism carries a slight asymmetry of Frobenius twists: the left side involves $R^{(1)}$ while the right side involves R. This asymmetry disappears in the Hodge-Tate comparison theorem (i.e., when we replace crystalline cohomology with prismatic cohomology in the preceding formulation).

2. Divided powers via δ-structures and consequences for prismatic envelopes

To relate crystalline and prismatic cohomology, we need to relate the construction of divided power envelopes (Construction 1.1) to a prismatic construction. This will ultimately follow from the next lemma, which gives the crucial relation between divided powers and δ-structures, and implies that the operation of adjoining all divided powers of an element to a δ-ring is a "finitely presented" operation on (a reasonably large subcategory of all) δ-rings.

[3]The naturality is not immediate from Theorem 1.6 as we had to choose a lift of R to A in this argument to invoke Theorem 1.6. However, one obtains the desired statement by simply imitating the proof of Theorem 1.6 while working exclusively mod p coefficients.

[4]The cohomology ring of a strictly commutative differential graded algebra is strictly graded commutative.

[5]The usual presentation of the Cartier map is slightly different: in degree i, it is given by "$\frac{\phi^*}{p^i}$", where ϕ denotes a lift of relative Frobenius to A/p^2. As the Bockstein involves division by p, it is not too difficult to see that this agrees with our construction in degree 1 and thus in all degrees by multiplicativity. The advantage of the formulation above is that it does not involve choosing lifts to A/p^2; the disadvantage is that it relies on crystalline cohomology.

Lemma 2.1 (Realizing divided powers via δ-structures)**.** *Let A be a p-torsionfree δ-ring. Let P be a δ-A-algebra that is p-completely flat over A. (For example, P could be ind-smooth as an A-algebra.) Let $J = (p, x_1, ..., x_r) \subset P$ be an ideal such that $x_1, ..., x_r \in P$ form a regular sequence on P/p. Then the δ-A-algebra $P\{\frac{\phi(J)}{p}\}^\wedge$ coincides with $D_J(P)$.*

In Construction 1.4, if we take P to be a free δ-A-algebra, then the ideals $J^n \subset P^n$ are filtered colimits of pairs $J \subset P$ satisfying the conditions in the above lemma.

Proof sketch. We only explain the proof when $P = \mathbf{Z}_p\{x\}$ with $J = (p, x)$; the general case is deduced by a base change argument that we omit.

It is enough to check that the δ-P-algebra $P\{\frac{\phi(x)}{p}\} := P\{y\}/(py - \phi(x))_\delta$ obtained by freely adjoining $\frac{\phi(x)}{p}$ is p-torsionfree, and coincides with the subring $D := P[\{\frac{x^n}{n!}\}_{n\geq 1}] \subset P[1/p]$.

For p-torsionfreeness, note that we have a pushout diagram

$$\begin{tikzcd} \mathbf{Z}_p\{z\} \arrow[r] \arrow[d, "z \mapsto \phi(x)"] & \mathbf{Z}_p\{z, y\}/(py - z)_\delta \simeq \mathbf{Z}_p\{y\} \arrow[d, "y \mapsto \frac{\phi(x)}{p}"] \\ P := \mathbf{Z}_p\{x\} \arrow[r] & P\{\frac{\phi(x)}{p}\} \end{tikzcd}$$

of δ-$\mathbf{Z}_p$-algebras. The left vertical map is abstractly the Frobenius endomorphism of $\mathbf{Z}_p\{z\}$; one can check that this map is flat (argument omitted). Consequently, the right vertical map is also flat, and thus $P\{\frac{\phi(x)}{p}\}$ is indeed p-torsionfree as $\mathbf{Z}_p\{y\}$ is so.

Inverting p in the above pushout diagram also shows that $P[1/p] \simeq P\{\frac{\phi(x)}{p}\}[1/p]$, so we can view $P\{\frac{\phi(x)}{p}\}$ as a subring of $P[1/p]$. Moreover, $P\{\frac{\phi(x)}{p}\}$ identifies with $P\{\frac{x^p}{p}\}$ as $\phi(x) = x^p + p\delta(x)$. Thus, $P\{\frac{\phi(x)}{p}\}$ coincides with the smallest δ-subring of $P[1/p]$ containing P and $\frac{x^p}{p}$, i.e., $C := P[\{\delta^i(\frac{x^p}{p})\}_{i\geq 1}]$ of $P[1/p]$. We shall show that $C = D$ as subrings of $P[1/p]$.

To show $D \subset C$, we must check that $\gamma_n(x) = \frac{x^n}{n!} \in C$ for all $n \geq 1$. For any $z \in P[1/p]$, a computation of p-adic valuations of factorials shows that $\gamma_n(\gamma_p(z)) = u \cdot \gamma_{np}(z)$ where $u \in \mathbf{Z}_{(p)}^*$. Since we already have $\gamma_p(x) \in C$ by definition, a small induction on n (omitted) reduces us to showing the following general statement:

($*$) If E is a p-torsionfree δ-ring and $z \in E$ with $\gamma_p(z) \in E$, we have $\gamma_{p^2}(z) \in E$.

To prove ($*$), we compute

$$\delta(\frac{z^p}{p}) = \frac{1}{p}\big(\phi(\frac{z^p}{p}) - \frac{z^{p^2}}{p^p}\big) = \frac{(z^p + p\delta(z))^p}{p^2} - \frac{z^{p^2}}{p^{p+1}}.$$

The second term on the right coincides with $\gamma_{p^2}(z)$ up to a p-adic unit, so it suffices to prove that the first term on the right lies in E. But $z^p + p\delta(z) \in pE$ by the assumption $\gamma_p(z) \in E$, so $(z^p + p\delta(z))^p \in p^p E \subset p^2 E$, so the first term on the right does indeed lie in E, as wanted.

To show $C \subset D$, it suffices to show that the endomorphism ϕ of P defines an endomorphism ϕ of D that is a lift of Frobenius on D/p (and thus a δ-structure). Note that ϕ obviously extends to an endomorphism of $P[1/p]$, so $\phi(\gamma_n(x))$ makes sense in $P[1/p]$ for all $n \geq 1$. Since D is generated over P by the divided powers $\gamma_n(x)$ of x, it is enough to show that $\phi(\gamma_n(x)) - \gamma_n(x)^p \in pD$ for all $n \geq 1$: the containment in D will show that ϕ extends to an endomorphism of D, and then the containment in pD will show that ϕ is a lift of Frobenius. In fact, since $\gamma_{np}(x) = \gamma_p(\gamma_n(x))u$ for a p-adic unit u, we have $\gamma_n(x)^p \in pD$, so it is enough to show that $\phi(\gamma_n(x)) \in pD$ as well. This is an elementary calculation and we omit it in these notes. $\square$

Remark 2.2. Lemma 2.1 is highly sensitive to the choice of the distinguished element "p" as the denominator. Indeed, if $(A,(d))$ is a general bounded prism and $P = A[x]$ with $\delta(x) = 0$, then the derived (p,d)-complete ring $P\{\frac{\phi(x)}{d}\}^\wedge$ obtained by freely adjoining $\frac{\phi(x)}{d}$ is quite different from the derived (p,d)-completed divided power envelope of (x) in P. In fact, when $(A,(d)) = (\mathbf{Z}_p[\![q-1]\!], ([p]_q))$ from Example III.1.2 (2), we shall later see that $P\{\frac{\phi(x)}{[p]_q}\}^\wedge$ coincides with the *q-divided power envelope* of (x) in P. This is ultimately the reason that prismatic cohomology differs from crystalline cohomology when working over a prism $(A,(d))$ that is not crystalline.

Using Lemma 2.1, we can also fulfil a promise made in Lecture V: we obtain a relatively explicit description (at least one that is does not involve transfinite constructions) of the prismatic envelopes encountered when computing prismatic cohomology of smooth algebras.

Corollary 2.3. *Let $(A,(d))$ be a bounded prism. Let P be a (p,d)-completely flat δ-A-algebra, and let $x_1, ..., x_r \in P$ be a sequence which is (p,d)-completely regular relative to A[6]. Set $J = (p,d,x_1,...,x_r)$.*

(1) The derived (p,d)-completion E of the δ-ring obtained by freely adjoining $\frac{x_i}{d}$ to P in the category of δ-rings is (p,d)-completely flat over A.

(2) The ring E from (1) coincides with the prismatic envelope $P\{\frac{J}{d}\}^\wedge$ (Lemma V.5.1).

(3) The prismatic envelope $P\{\frac{J}{d}\}^\wedge$ is (p,d)-completely flat over A, and its formation commutes with base change along maps of bounded prisms.

Proof sketch. It is enough to show (1). Indeed, flatness implies that E is d-torsionfree (by a variant of Lemma III.2.4), so $(P,J) \to (E,(d))$ is a map of δ-pairs over $(A,(d))$ with the target being a prism. It is easy to check using the definition of E that this map has the desired universal property of $(P,J) \to (P\{\frac{J}{d}\}^\wedge, (d))$, giving (2). Finally, the base change compatibility is clear from the construction of E.

The statement that E is (p,d)-completely flat over A is proven in multiple steps. First, when $(d) = (p)$, this is proven using Lemma 2.1 and standard properties of divided power algebras. Next, if $d = \phi(d')$, one reduces to the case $(d) = (p)$ via base change along $A \to A\{\frac{\phi(d')}{p}\}^\wedge$ (which is itself controlled by the previous case) and the irreducibility lemma for distinguished elements (Lemma III.1.7); strictly speaking, this reduction only works when d is a nonzerodivisor modulo p and in general we work with simplicial commutative δ-rings. Finally, the general case follows as any element of any δ-ring lies in the image of ϕ after a faithfully flat base change as the Frobenius on the free δ-ring $\mathbf{Z}_p\{x\}$ is flat. □

3. The crystalline comparison for prismatic cohomology

Fix a bounded prism $(A,(d))$ and a formally smooth $A/(d)$-algebra R. Recall the following Cech-Alexander type construction of prismatic cohomology from Lecture V (Construction V.5.3).

Construction 3.1 (Computing prismatic cohomology)**.** Choose a free δ-A-algebra P equipped with a surjection $P \to R$. Let $P^\bullet$ be the Cech nerve of $A \to P$, so each P^n is a free δ-A-algebra. Moreover, we have an induced surjection $P^n := P^{\otimes_A (n+1)} \xrightarrow{\mu} P \to R$ where μ is the multiplication map. These maps are compatible as n varies, so taking kernels gives a cosimplicial

[6] This terminology is nonstandard. For us, it means that the map $\mathrm{Kos}(A;p,d) \to \mathrm{Kos}(P;p,d,x_1,...,x_r)$ is a flat map of simplicial commutative rings. If $(p) = (d)$ or (p,d) is a regular sequence of length 2, this is equivalent to saying that the Koszul complex $\mathrm{Kos}(P/(p,d);x_1,...,x_r)$ has homology only in degree 0 where it is given by a flat $A/(p,d)$-module.

ideal $J^\bullet \subset P^\bullet$. Applying the construction of prismatic envelopes (Lemma V.5.1), we obtain a cosimplicial p-torsionfree δ-A-algebra

$$C^\bullet_{\Delta}(R/A) := \Big(P^0\{\tfrac{J^0}{d}\}^\wedge \rightrightarrows P^1\{\tfrac{J^1}{d}\}^\wedge \overset{\rightarrow}{\rightrightarrows} P^2\{\tfrac{J^2}{d}\}^\wedge \rightrightarrows\rightrightarrows \cdots \Big). \tag{3}$$

By Construction V.5.3, we have $\Delta_{R/A} \simeq \mathrm{Tot}(C^\bullet_{\Delta}(R/A))$. As in Construction 1.4, we can make the above construction functorial in the smooth A/p-algebra R by choosing P functorially in R; for example, we may take P to be the free δ-A-algebra on $W(R)$, with the map $P \to R$ determined as the composition $P \to W(R) \to R$ where the second map is the canonical map while the first map is the δ-map determined by using the δ-A-algebra structure on $W(R)$ (coming via right adjointness of $W(-)$ to the forgetful functor from δ-rings to rings) and sending each generator in P to the corresponding element of $W(R)$.

Assume from now that $(d) = (p)$. Our goal is to prove the following theorem.

Theorem 3.2 (The crystalline comparison)**.** *There is a canonical isomorphism*

$$(\phi_A^* \Delta_{R/A})^\wedge \simeq R\Gamma_{crys}(R/A)$$

of commutative algebras in $D(A)$ compatibly with Frobenius.

Let us first explain the idea for constructing the map $\Delta_{R/A} \to \phi_* R\Gamma_{crys}(R/A)$ informally. Following the notation in Construction 1.4, set $(D^\bullet, K^\bullet) := (D_{P^\bullet}(J^\bullet), \ker(D_{P^\bullet}(J^\bullet) \to R))$. So each D^n is a p-torsionfree A-algebra, each $K^n \subset D^n$ is a divided power ideal, and $R\Gamma_{crys}(R/A)$ is computed by totalizing $D^\bullet$. Now $D^\bullet$ lifts to a diagram of δ-A-algebras by Lemma 2.1 provided we take input ring P itself to be a free δ-A-algebra. By the existence of divided powers on $K^\bullet$, we have the crucial containment

$$\phi(K^\bullet) \subset pD^\bullet$$

since for any $x \in K^n$, we have $\phi(x) = x^p + p\delta(x) = p \cdot ((p-1)!\gamma_p(x) + \delta(x)) \in pD^n$. Thus, we obtain a commutative diagram

$$\begin{tikzcd} D^\bullet \arrow[r, "\phi"] \arrow[d] & \phi_* D^\bullet \arrow[d] \\ D^\bullet/K^\bullet \simeq R \arrow[r] & \phi_* D^\bullet/p. \end{tikzcd}$$

Ignoring the top left vertex, this gives a diagram in $(R/A)_{\Delta}$, and consequently we obtain a canonical map

$$\Delta_{R/A} \to \mathrm{Tot}(\phi_{A,*} D^\bullet) \simeq \phi_{A,*} R\Gamma_{crys}(R/A)$$

of commutative algberas in $D(A)$. The adjoint to this map is the one in Theorem 3.2.

Proof sketch. For simplicity, we only give a proof when $A = \mathbf{Z}_p$. In this case, the map ϕ_A is the identity, so we must show that $\Delta_{R/A} \simeq R\Gamma_{crys}(R/A)$. We shall prove this by comparing the cosimplicial ring (1) from Construction 1.4 with the analogous construction (3) for prismatic cohomology. Our proof actually gives a homotopy equivalence of these two cosimplicial rings.

As free δ-A-algebras are ind-smooth over A, Construction 3.1 is compatible with Construction 1.4, i.e., we may use the same surjection $P \to R$ for both constructions. Writing $P^\bullet$ for the Cech nerve of $A \to P$ and $J^\bullet \subset P^\bullet$ for the kernel of the augmentation $P^\bullet \to P \to R$, it suffices to give a homotopy equivalence between the cosimplicial A-algebras

$$P^\bullet\{\frac{J^\bullet}{p}\}^\wedge \quad \text{and} \quad D_{J^\bullet}(P^\bullet).$$

Note that the map $A \to P^\bullet$ is a homotopy equivalence of cosimplicial δ-A-algebras by Cech theory. In particular, the map $\phi_{P^\bullet}$ is also a homotopy equivalence, so we obtain a homotopy equivalence

$$P^\bullet\{\frac{J^\bullet}{p}\}^\wedge \simeq \phi_{P^\bullet}^*(P^\bullet\{\frac{J^\bullet}{p}\})^\wedge \cong P^\bullet\{\frac{\phi(J^\bullet)}{p}\}^\wedge$$

of cosimplicial A-algebras (where the first map is the homotopy equivalence induced by p-completed base change along $\phi_{P^\bullet}$ while the second map is an isomorphism of cosimplicial rings obtained by reidentifying the terms). It now suffices to identify

$$P^\bullet\{\frac{\phi(J^\bullet)}{p}\}^\wedge \quad \text{and} \quad D_{J^\bullet}(P^\bullet)$$

as cosimplicial A-algebras, which follows from Lemma 2.1.

The phrase "compatibly with Frobenius" in Theorem 3.2 still needs to be justified. More precisely, the quasi-isomorphism constructed above matches the Frobenius on $\Delta_{R/A}$ with the Frobenius ϕ_1 on $R\Gamma_{crys}(R/A)$ coming from the δ-structure on $D_{J^\bullet}(P^\bullet)$ induced by the one on $P^\bullet$ via Lemma 2.1. However, the Frobenius on R itself induces (by functoriality) a Frobenius ϕ_2 on $R\Gamma_{crys}(R/A)$. We must thus identify ϕ_1 and ϕ_2. We leave this compatibility to the reader as an exercise in unwinding the definition of ϕ_2 in terms of the crystalline site. $\square$

Theorem 3.2 globalizes to an interesting Frobenius descent statement for the module underlying the Gauss-Manin connection; even though we have not discussed the global theory in these lectures, let us at least formulate the statement.

Corollary 3.3. *Let $f : X \to \mathrm{Spec}(A/p)$ be a proper smooth morphism. Then there is a canonical identification $\phi^* R\Gamma(X, \Delta_{X/A}) \simeq R\Gamma(X, \mathcal{O}_{X/A,crys})$ of commutative algebras in in $D(A)$. In particular, by reducing modulo p and using Theorem 1.8, we learn that the relative de Rham cohomology $R\Gamma(X, \Omega^*_{X/(A/p)}) \in D(A/p)$ has a canonical descent along the Frobenius on A/p.*

A similar Frobenius descent phenomenon in the context of Dieudonné modules of finite flat group schemes was already observed in [3, §7]. The Frobenius descent of the cohomology groups of $R\Gamma(X, \Omega^*_{X/(A/p)})$ also follows from [6], which realizes the descended modules as Higgs bundles.

4. The Hodge-Tate comparison

We end by sketching how to deduce Theorem 0.1 from Theorem 3.2 and base change arguments. Thus, assume that $(A, (d))$ is a bounded prism and R is a formally smooth $A/(d)$-algebra.

Sketch of proof of the Theorem 0.1. Our goal is to show that Hodge-Tate comparison maps

$$\eta_R^* : \Omega^*_{R/(A/d)} \to H^*(\overline{\Delta}_{R/A})$$

are isomorphisms R-modules for all i. The strategy of the proof is to deduce this in characteristic p case from the Cartier isomorphism and the crystalline comparison for prismatic cohomology; the general case is reduced to this case via (somewhat technical) base change arguments.

First assume $(d) = (p)$ as ideals of A and that the Frobenius $\phi_{A/p}$ on A/p is faithfully flat. In this case, one checks that $\phi^*_{A/p}(\eta_R^*)$ agrees with the Cartier isomorphism (Construction 1.9) under the isomorphisms coming from Theorem 3.2 (relating prismatic and crystalline cohomology) and Theorem 1.8 (relating crystalline cohomology moduo p to de Rham cohomology). In particular, $\phi^*_{A/p}(\eta_R^*)$ is an isomorphism, and thus each η_R^* is also an isomorphism as $\phi_{A/p}$ is faithfully flat.

To handle the general case where $(d) = (p)$, one first proves an étale localization property for prismatic cohomology; this allows us to reduce checking the desired statement when $R = A/p[x_1, ..., x_n]$ is the polynomial ring, in which case one can reduce via base change to the case $A = \mathbf{Z}_p$, which was covered above. We omit the details in these notes.

To handle the general case, it is convenient to formulate the statement of the theorem at the derived category level. As R is formally smooth over A/I, each $\Omega^i_{R/(A/d)}$ is a finite projective R-module. We may thus non-canonically choose a map

$$\eta_R : \bigoplus_i \Omega^i_{R/(A/d)}[-i] \to \overline{\Delta}_{R/A}$$

in $D(R)$ inducing the map η_R^* on cohomology. Our goal is to show that η_R is an isomorphism in $D(R)$. It is easy to see that this statement can be checked after (p,d)-completely faithfully flat base change on A. In particular, we may assume $d = \phi(e)$ for some $e \in A$.

It remains to prove η_R is an isomorphism. We only give the argument when d (or equivalently e) is a nonzerodivisor A/p; in conjunction with the characteristic p treated above, this covers most examples encountered in practice. Let $D := A\{\frac{d}{p}\}^\wedge \simeq A\{\frac{\phi(e)}{p}\}^\wedge$. By Lemma 2.1, this ring is p-torsionfree and coincides with $D_{(e)}(A)$, the p-adically completed divided power envelope of (e) in A. Consider the canonical map $\alpha : A \to D$. Then $\alpha(d) = pu$ for a unit u by the irreducibility lemma for distiguished elements (Lemma III.1.7), so $\alpha : A \to D$ can also be viewed as a map $(A,(d)) \to (D,(p))$ of bounded prisms. Moreover, this map factors modulo d as

$$A/d \to A/(p,d) = A/(p,e^p) \to D/p,$$

where the first map is the canonical one (and thus has finite Tor amplitude by our assumption that d is a nonzerodivisor on A/p), and the second one is faithfully flat by the structure of divided power algebras. It is then easy to see that p-completed base change along α is conservative on the category of derived (p,d)-complete complexes, and that it commutes with totalizations of cosimplicial derived (p,d)-complete A-modules. In particular, it follows from Construction 3.1 (and the base change compatibility in Corollary 2.3) that p-completed base change along α induces an isomorphism $\overline{\Delta}_{R/A}\widehat{\otimes}^L_A D \simeq \overline{\Delta}_{R\widehat{\otimes}_A D/D}$. As the analogous property also holds true for differential forms, we learn that p-completed base change along α carries η_R to an isomorphism (by the case $(d) = (p)$ treated above). As this base change functor is conservative on the relevant category of complexes, we conclude that η_R must itself be an isomorphism. ☐

Thanks to the Hodge-Tate comparison and sheaf property for differential forms, one can globalize prismatic cohomology to arbitrary formally smooth schemes.

Corollary 4.1 (Globalization of prismatic cohomology)**.** *Let X be any formally smooth $A/(d)$-scheme. There exists a functorially defined (p,d)-complete commutative algebra object $\Delta_{X/A} \in D(X,A)$ equipped with a ϕ_A-linear self-map $\phi_X : \Delta_{X/A} \to \Delta_{X/A}$ with the following features:*

(1) For any affine open $U := \mathrm{Spf}(R) \subset X$, there is a natural isomorphism between $R\Gamma(U, \Delta_{X/A})$ and $\Delta_{R/A}$ of p-complete commutative objects in $D(A)$ that carries ϕ_X to ϕ_R.

(2) Set $\overline{\Delta}_{X/A} := \Delta_{X/A} \otimes^L_A A/(d) \in D(X, A/I)$. Then $\overline{\Delta}_{X/A}$ is naturally an object of $D_{perf}(X)$, and we have canonical isomorphisms $\Omega^i_{X/A} \simeq H^i(\overline{\Delta}_{X/A})$ for all i compatible with those coming from Theorem 0.1 under the isomorphisms in (1).

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