LECTURE V: THE PRISMATIC SITE

In this lecture, we define the prismatic site of a scheme, and formulate the Hodge-Tate comparison theorem relating the associated prismatic cohomology to differential forms.

1. The basic setup

We shall fix a "base" prism (A, I) as well as a formally smooth[1] A/I-algebra R. Recall (Theorem I.3.3) that the goal of prismatic cohomology is to produce a complex $\Delta_{R/A}$ of A-modules with a "Frobenius" endomorphism $\phi_{R/A}$ such that the following hold true:

- $\Delta_{R/A} \otimes^L_A A/I$ is related to differential forms on R (relative to A/I).
- The pair $(\Delta_{R/A}[\frac{1}{I}]^\wedge, \phi_{R/A})$ is related to the p-adic étale cohomology of $R[1/p]$.

In other words, we aim to construct a deformation, parametrized by $\mathrm{Spec}(A)$, relating the algebraic de Rham cohomology of $R/(A/I)$ to the p-adic étale cohomology $R[1/p]$.

To focus on the essential ideas, we shall often impose extra assumptions on our base prism (A, I), especially when it simplifies the exposition. In particular, we shall always assume (A, I) is bounded and often that $I = (d)$ is principal. Let us recall the key examples that we aim to capture.

(1) Let S be a perfect $\mathbf{F}_p$-algebra. Set $A = W(S)$. Then $(A, (p))$ is a bounded prism, and in this case our theory will be equivalent to crystalline cohomology. More generally, allowing A to be possibly imperfect but maintaining $I = (p)$, our theory will give a Frobenius descent of Grothendieck's crystalline cohomology.

(2) Let W be the ring of Witt vectors of a perfect field of characteristic p. Let $E(u) \in W[\![u]\!]$ be an Eisenstein polynomial (i.e., the polynomial is monic, the constant term has p-adic valuation 1, and the remaining terms have p-adic valuation > 1); the case $W = \mathbf{Z}_p$ and $E(u) = u - p$ corresponds to the example in Theorem I.3.3. We endow $A := W[\![u]\!]$ with a δ-structure by requiring $\delta(u) = 0$ (and thus ϕ is the unique lift of Frobenius on A sending u to u^p). Then $(A, (E(u)))$ is a bounded prism, and the prismatic theory in this case yields a cohomological lift of the notion of Breuil-Kisin modules [3].

(3) Let $A = \mathbf{Z}_p[\![q-1]\!]$, viewed as a δ-ring via $\phi(q) = q^p$. If we write $[p]_q = \frac{q^p-1}{q-1} = 1+q+...+q^{p-1}$, then $(A, [p]_q)$ is a bounded prism. The prismatic cohomology theory constructed in this case is related to the q-deformation of Rham cohomology discussed in Remark I.3.4 (f) and considered in [5].

(4) Let A_{inf} be the $(p, [p]_q)$-adic completion of the perfection $\mathrm{colim}_\phi A$ of the ring A from Example (3). Then the pair $(A_{\mathrm{inf}}, ([p]_q))$ is a bounded perfect prism, and the corresponding prismatic cohomology theory will recover the one constructed in [1]. More generally, if S is any p-torsionfree perfectoid ring in the sense of Lecture IV, then the perfect prism $(A_{\mathrm{inf}}(S), \ker(\theta_S))$ is bounded with $\ker(\theta_S)$ principal.

Fix a bounded prism (A, I) and a formally smooth A/I-algebra R for the rest of the lecture. We also assume that $I = (d)$ is principal, though we try to use this assumption sparingly.

[1] As this term is potentially confusing, let us clarify what it means for our purposes. If $\overline{A}$ is a p-complete ring, then a formally smooth $\overline{A}$-algebra is given by a p-complete $\overline{A}$-algebra R such that $\overline{A}/p \to R \otimes^L_{\overline{A}} \overline{A}/p$ has homology concentrated in degree 0 where it is given by a smooth $\overline{A}/p$-algebra. Thus, this notion could also be called *p-complete smoothness* in analogy with p-complete flatness (Definition III.2.5). A typical example is given by the p-completion of a smooth $\overline{A}$-algebra (provided $\overline{A}$ has bounded p^∞-torsion, as we shall assume).

2. The prismatic site: definition

Prismatic cohomology of R will be defined by "probing" R with prisms over (A, I). This intuitive idea is formalized in the following definition.

Definition 2.1 (The prismatic site)**.** The *prismatic site of R relative A*, denoted $(R/A)_{\mathbb{\Delta}}$, is the category whose objects are given by prisms (B, IB) over (A, I) together with an A/I-algebra map $R \to B/IB$; morphisms in this category in the obvious way. We shall write a typical object of this category as $(R \to B/IB \leftarrow B)$ and display it as

$$\begin{tikzcd} A \arrow[rr] \arrow[d] & & B \arrow[d] \\ A/I \arrow[r] & R \arrow[r] & B/IB. \end{tikzcd} \tag{1}$$

We endow $(R/A)_{\mathbb{\Delta}}$ with the indiscrete Grothendieck topology, so all presheaves are sheaves. Write $\mathcal{O}_{\mathbb{\Delta}}$ and $\overline{\mathcal{O}}_{\mathbb{\Delta}}$ for the functors on this category defined by sending $(R \to B/IB \leftarrow B) \in (R/A)_{\mathbb{\Delta}}$ to B and B/IB respectively; the functor $\mathcal{O}_{\mathbb{\Delta}}$ is naturally valued in (p, I)-complete δ-A-algebra, while $\overline{\mathcal{O}}_{\mathbb{\Delta}}$ is valued in p-complete R-algebras. Note that $\overline{\mathcal{O}}_{\mathbb{\Delta}} \simeq \mathcal{O}_{\mathbb{\Delta}}/I\mathcal{O}_{\mathbb{\Delta}}$.

Remark 2.2. Strictly speaking, the category defined in Definition 2.1 site is the *opposite* of what ought to be called the prismatic site. We shall indulge in this abuse of notation repeatedly in the sequel as there does not seem to be any potential for confusion. In particular, we abusively refer to the covariant functors $\mathcal{O}_{\mathbb{\Delta}}$ and $\overline{\mathcal{O}}_{\mathbb{\Delta}}$ as sheaves on $(R/A)_{\mathbb{\Delta}}$.

Remark 2.3. If X is a formally smooth A/I-scheme, there is an evident analog of Definition 2.1 yielding the prismatic site of $(X/A)_{\mathbb{\Delta}}$. In these lectures, we only explain the theory in the affine case, i.e., when $X := \mathrm{Spf}(R)$. There are two reasons to do this:

(1) Not much generality is lost: once the relevant comparison theorems are proven in the affine case in a sufficiently functorial fashion, globalization is relatively straightforward. On the other hand, the notation becomes simpler as we can work with rings everywhere.

(2) We can avoid discussing the Grothendieck topology on $(R/A)_{\mathbb{\Delta}}$, i.e., we can just work with presheaves to get the "correct" answers. This is similar to the corresponding phenomenon in crystalline cohomology: when defining the crystalline cohomology of an affine scheme, one may just work with the indiscrete topology on the crystalline site of the affine (so all presheaves are sheaves) while still computing the correct crystalline cohomology groups.

Remark 2.4. Definition 2.1 evidently makes sense for all A/I-algebras, not just the formally smooth ones. However, if one drops all assumptions on R, the resulting cohomology theory seems essentially incomputable. This is analogous to the situation for crystalline cohomology where one can only really compute the theory for lci rings.

Remark 2.5 (Relation to convergent cohomology of [4])**.** Assume that A is the ring of Witt vectors of a perfect field k of characteristic p, and $I = (p)$. In this case, the objects of $(R/A)_{\mathbb{\Delta}}$ are closely related to *p-adic enlargements* of $\mathrm{Spec}(R)$ in the sense of [4, Definition 2.1]. More precisely, the latter is given by a diagram of commutative rings of the form (1) where B is a p-torsionfree and p-adically complete ring. In particular, forgetting the δ-structure defines a functor from $(R/A)_{\mathbb{\Delta}}$ to the category of p-adic enlargements of $\mathrm{Spec}(R)$. However, since we have forgetten the Frobenius lifts, the behaviour of coproducts (and thus also of cohomology) in $(R/A)_{\mathbb{\Delta}}$ differs from that in the category of p-adic enlargements of $\mathrm{Spec}(R)$.

Let us describe some examples of objects.

Example 2.6. Assume $A/I = R$. In this case, the category $(R/A)_{\Delta}$ identifies with the category of prisms over (A, I) and thus has an initial object given by $(R \simeq A/I \leftarrow A)$.

Example 2.7. Say $R = A/I\langle X\rangle$ is the p-adic completion of $A/I[X]$. The category $(R/A)_{\Delta}$ does not admit an initial object in this case. However, there are some natural objects that will be relevant later. For example, let $A[X]$ be the polynomial ring over A, viewed as a δ-A-algebra via $\delta(X) = 0$. Let B be the (p, I)-completion of $A[X]$. Then $B/IB \simeq R$, so the resulting diagram $(R \simeq B/IB \leftarrow B)$ gives an object of $(R/A)_{\Delta}$.

More generally, one can show that for any formally smooth A/I-algebra R, there exists a (p, I)-completely smooth lift $\widetilde{R}$ of R to A together with a δ-A-algebra structure on $\widetilde{R}$. In this case, we obtain the object $(R \simeq \widetilde{R}/I\widetilde{R} \leftarrow \widetilde{R}) \in (R/A)_{\Delta}$.

Remark 2.8 (The perfect prismatic site)**.** Assume (A, I) is a perfect prism. Using the equivalence between perfectoid rings and perfect prisms (Theorem IV.2.3), one checks the following: for any map $R \to S$ with S perfectoid, the composite $A/I \to R \to S$ lifts to a unique map $(A, I) \to (A_{\mathrm{inf}}(S), \ker(\theta_S))$ of prisms. In particular, any such S determines an object $(R \to S \leftarrow A_{\mathrm{inf}}(S)) \in (R/A)_{\Delta}$. Summarizing, this construction defines a functor from the category of perfectoid R-algebras to $(R/A)_{\Delta}$. Unwinding definitions shows that this functor is fully faithful with essential image consisting of those $(R \to B/IB \leftarrow B) \in (R/A)_{\Delta}$ with (B, IB) being a perfect prism. We shall sometimes write $(R/A)_{\Delta}^{\mathrm{perf}}$ for this essential image, and refer to it as the *perfect prismatic site*. If one further restricts attention to perfectoid R-algebras S with S integrally closed in $S[1/p]$, one obtains (up to change of language) the *diamond* of $\mathrm{Spa}(R[1/p], R)$ in the sense of [6].

Example 2.9. Assume (A, I) is a perfect prism. Let us give an explicit example of an object of $(R/A)_{\Delta}^{\mathrm{perf}}$. Say $R = A/I\langle X\rangle$ as in Example 2.7. Take $S := A/I\langle X^{\frac{1}{p^\infty}}\rangle$. Using the uniqueness assertion in Theorem IV.2.3, one checks that the ring $A_{\mathrm{inf}}(S)$ is given by the (p, I)-completion of the δ-A-algebra $A[X^{\frac{1}{p^\infty}}]$, where the δ-structure is determined by requiring each $X^{\frac{1}{p^n}}$ have rank 1. There is an obvious A/I-algebra $R \to S$, giving an object $(R \to S \leftarrow A_{\mathrm{inf}}(S)) \in (R/A)_{\Delta}^{\mathrm{perf}}$.

Note that the map $R \to S$ constructed above is p-completely faithfully flat. More generally, one can show that for any formally smooth A/I-algebra R, there exists a p-completely faithfully flat map $R \to S$ with S perfectoid. This is one reason why the study of perfectoid rings has implications for prismatic cohomology.

Prismatic cohomology is defined immediately from Definition 2.1:

Definition 2.10 (Prismatic cohomology)**.** The *prismatic complex* $\Delta_{R/A}$ of R is defined to be the cohomology of the sheaf $\mathcal{O}_{\Delta}$, i.e., $\Delta_{R/A} := R\Gamma((R/A)_{\Delta}, \mathcal{O}_{\Delta})$. This is a (p, I)-complete commutative algebra object in $D(A)$; the Frobenius action on $\mathcal{O}_{\Delta}$ induces a ϕ-semilinear map $\Delta_{R/A} \to \Delta_{R/A}$.

Similarly, define the *Hodge-Tate complex* $\overline{\Delta}_{R/A} := R\Gamma((R/A)_{\Delta}, \overline{\mathcal{O}}_{\Delta})$. This is a p-complete commutative algebra object in $D(R)$, and we have $\Delta_{R/A} \otimes_A^L A/I \simeq \overline{\Delta}_{R/A}$.

We shall later explain how to compute cohomology of a presheaf explicitly on a category satisfying some mild hypothesis. In the next section, we explain how to identify the cohomology groups of the Hodge-Tate complex. For now, we simply mention the following example.

Example 2.11. If $R = A/I$, then $\Delta_{R/A} \simeq A$ and $\overline{\Delta}_{R/A} \simeq A/I$. Indeed, this follows immediately from the definitions as $(R/A)_{\Delta}$ has an initial object $(R \simeq A/I \leftarrow A) \in (R/A)_{\Delta}$.

3. The Hodge-Tate comparison theorem

Our next goal is to formulate the Hodge-Tate comparison theorem, which gives a relation between the Hodge-Tate complex $\overline{\Delta}_{R/A}$ and differential forms on R relative to A/I. For this, we shall need a well-known universal property of the de Rham complex, which we recall next.

3.1. **Reminders on the de Rham complex.** Let us first recall standard conventions for commutative differential graded algberas.

Notation 3.1. Let B be a commutative ring. A differential graded algebra over B (or B-dga for short) is given by a pair (E^*, d) where E^* is a graded B-algebra and $d : E^* \to E^{*+1}$ is a B-linear map satisfying the signed Leibnitz rule. We say that such a pair (E^*, d) is *graded commutative* if the graded ring E^* is graded commutative, i.e., $a \cdot b = (-1)^{\deg(a)\deg(b)} b \cdot a$ for homogeneous elements a and b. If in addition we also have $a^2 = 0$ for elements a of odd degree, we call the pair (E^*, d) *strictly graded commutative*. Note that this distinction matters only in the presence of 2-torsion.

We can now construct the de Rham complex and formulate its universal property.

Construction 3.2 (The de Rham complex and its universal property)**.** Let $B \to C$ be a map of commutative rings. As usual, we denote its algebraic de Rham complex by

$$(\Omega^*_{C/B}, d_{dR}) := \Big(C \to \Omega^1_{C/B} \to \Omega^2_{C/B} \to \cdots\Big),$$

viewed as a genuine chain complex (and not merely in the derived category). Taking wedge product of differential forms turns the above complex into a strictly graded commutative B-dga. This construction has the following universal property:

Lemma 3.3. *Let (E^*, d) be a graded commutative B-dga with $E^i = 0$ for $i < 0$. Assume that we are given a B-algebra map $\eta : C \to E^0$ such that for every $x \in C$, the element $y := d(\eta(x)) \in E^1$ satisfies $y \cdot y = 0$. (This last condition is automatic if E^* is strictly graded commutative.) Then the map $C \to E^0$ extends uniquely to a map $\Omega^*_{C/B} \to E^*$ of B-dgas.*

Proof sketch. As E^* is a commutative graded ring, we can regard each E^i as a module over the commutative ring E^0 and thus also over C via η. The composite map $C \xrightarrow{\eta} E^0 \xrightarrow{d} E^1$ is a B-linear derivation, so it extends to a map $\eta^1 : \Omega^1_{C/B} \to E^1$ by the universality of the de Rham derivation $C \to \Omega^1_{C/B}$. The strict graded commutativity of E^* and the universal property of the exterior algebra ensure that taking wedge powers η^1 gives a map $\eta^i : \Omega^i_{C/B} \to E^i$. We leave it to the reader to check that this gives the desired extension. □

Remark 3.4. In the sequel, we shall work with the continuous de Rham complex rather than the algebraic one, i.e., for our formally smooth A/I-algebra R, we abuse notation to set $\Omega^i_{R/(A/I)}$ to be the derived p-completion (as modules) of the usual R-module of i-forms on R relative to A. It is an exercise in using the formal smoothness of $A \to R$ and the boundedness of $A[p^\infty]$ to see that this R-module is finite projective of the correct rank, and that it coincides with the derived p-completion (as complexes) of the usual module of i-forms. By the universal property of derived p-completion, there is an evident variant of Lemma 3.3 in this setting provided the target B-dga is derived p-complete.

3.2. **Constructing the Hodge-Tate comparison maps.** We now use the above property of the de Rham complex to construct a canonical map from differential forms to the cohomology of the Hodge-Tate complex.

Construction 3.5 (The Hodge-Tate comparison maps)**.** Choose a generator $d \in I$. As $\overline{\Delta}_{R/A}$ is a commutative algebra object in $D(R)$, generalities on cohomology imply that $H^*(\overline{\Delta}_{R/A})$ is a graded commutative R-algebra; write $\eta : R \to H^0(\overline{\Delta}_{R/A})$ for the structure morphism. Note that we have exact sequences

$$0 \to \mathcal{O}_{\Delta}/d \xrightarrow{(-)\cdot d} \mathcal{O}_{\Delta}/d^2 \xrightarrow{can} \mathcal{O}_{\Delta}/d \to 0$$

of sheaves on $(R/A)_{\Delta}$. Taking cohomology, we obtain a Bockstein map

$$\beta_d : H^i(\overline{\Delta}_{R/A}) \to H^{i+1}(\overline{\Delta}_{R/A}).$$

A standard argument shows that this map gives a derivation on $H^*(\overline{\Delta}_{R/A})$. Thus, we have constructed graded commutative A/I-dga $(H^*(\overline{\Delta}_{R/A}), \beta_d)$ together with a map $\eta : R \to H^0(\overline{\Delta}_{R/A})$ of A/I-algebras. The following lemma shall be proven later (and is only non-trivial if $p = 2$):

Lemma 3.6. *The graded commutative A/I-dga $(E^*, d) := (H^*(\overline{\Delta}_{R/A}), \beta_d)$ together with the map $\eta : R \to H^0(\overline{\Delta}_{R/A})$ satisfies the condition from Lemma 3.3.*

Consequently, Lemma 3.3 ensures that the map η extends to a map

$$\eta_R^* : (\Omega^*_{R/(A/I)}, d_{dR}) \to (H^*(\overline{\Delta}_{R/A}), \beta_d)$$

of A/I-dgas. We call the maps η_R^* the *Hodge-Tate comparison maps.*

Remark 3.7 (A choice free formulation)**.** In Construction 3.5, we made a choice of a generator $d \in I$. Let us explain how to formulate the construction in a more invariant fashion (that makes sense for any base prism (A, I), not just the reasonable ones). For any A/I-module M and integer n, we define the *Breuil-Kisin twist* via $M\{n\} := M \otimes_{A/I} (I/I^2)^{\otimes n}$; note that I/I^2 is an invertible A/I-module, so $(I/I^2)^{\otimes n}$ has a natural meaning for any $n \in \mathbf{Z}$. Redoing Construction 3.5 then endows the graded A/I-module $H^*(\overline{\Delta}_{R/A})\{*\}$ with the structure of a graded commutative A/I-dga via the Bockstein differential β_I, and the analog of Lemma 3.6 gives a comparison map

$$\eta_R^* : (\Omega^*_{R/(A/I)}, d_{dR}) \to (H^*(\overline{\Delta}_{R/A})\{*\}, \beta_I)$$

of A/I-dgas.

The basic result for prismatic cohomology is the following:

Theorem 3.8 (The Hodge-Tate comparison theorem)**.** *The Hodge-Tate comparison map*

$$\eta_R^* : (\Omega^*_{R/(A/I)}, d_{dR}) \to (H^*(\overline{\Delta}_{R/A}), \beta_d)$$

is an isomorphism. In particular, we have $\Omega^i_{R/(A/I)} \simeq H^i(\overline{\Delta}_{R/A})$ for all i, so the complex $\overline{\Delta}_{R/A} \in D(R)$ is a perfect complex.

When $I = (p)$, we shall show later that $\Delta_{R/A}$ gives a Frobenius descent of crystalline cohomology $R\Gamma_{crys}(R/A)$; under this isomorphism, Theorem 3.8 corresponds to the Cartier isomorphism (except there is no spurious Frobenius twist). We can also see Theorem 3.8 explicitly in the example of a torus provided we use the presentation for $\Delta_{R/A}$ in terms of q-de Rham complexes from Lecture I.

Example 3.9 (Hodge-Tate isomorphism via the q-de Rham complex)**.** Let $(A, I) = (\mathbf{Z}_p[\![q - 1]\!], ([p]_q))$, so the map $q \mapsto \epsilon_p$ identifies A/I with $\mathbf{Z}_p[\epsilon_p]$, the ring of integers of $\mathbf{Q}_p(\epsilon_p)$, where ϵ_p is a primitive p-th root of 1. Let $R = A/I\langle X^{\pm 1}\rangle$ be the p-adic completion of $A/I[X^{\pm 1}]$, so we can write

$$R := \widehat{\bigoplus_{i \in \mathbf{Z}}} (A/I) \cdot X^i$$

as the displayed p-completed direct sum. In this case, as stated in Remark I.3.4 (f) (and we shall prove later) that $\Delta_{R/A}$ is computed by the following complex

$$\Delta_{R/A} \simeq \Big(A\langle X^{\pm 1}\rangle \xrightarrow{\nabla_q} A\langle X^{\pm 1}\rangle \frac{dX}{X} \Big) \simeq \widehat{\bigoplus_{i \in \mathbf{Z}}} \Big(A \cdot X^i \xrightarrow{[i]_q} A \cdot X^i \frac{dX}{X} \Big),$$

where $[i]_q := \frac{q^i - 1}{q - 1}$ is the q-analog of i and the completion if $(p, [p]_q)$-adic. Note that $[i]_q$ is invertible in A/I when i is not divisible by p: it is invertible modulo $q - 1$ and hence also in the residue field of A/I. Moreover, if $i = kp$ is divisible by p, then $[i]_q = 0$ in A/I as $\frac{q^p-1}{q-1}$ divides $\frac{q^{kp}-1}{q-1}$ for all k. Thus, the above identification reduces modulo $[p]_q$ to give a quasi-isomorphism

$$\overline{\Delta}_{R/A} \simeq \widehat{\bigoplus_{k\in\mathbf{Z}}}\Big((A/I)\cdot X^{kp} \xrightarrow{0} (A/I)\cdot X^{kp}\Big).$$

Thus, we see that $\overline{\Delta}_{R/A}$ has the shape predicted by Theorem 3.8: it has 2 nonzero cohomology groups, each of which is a free R-module of rank 1 (modulo unwrapping various identifications).

4. Reminder on cohomology of categories

The general theory of cohomology of sites takes on a fairly simple form when the topology on the site is the indiscrete topology (i.e., when all presheaves are sheaves). As this will be the case for the prismatic site, let us recall the definition.

Definition 4.1. Let $\mathcal{C}$ be a small category, and write $\mathrm{PShv}(\mathcal{C})$ for the corresponding presheaf topos. For any abelian presheaf F on $\mathcal{C}$, we define $R\Gamma(\mathcal{C}, F)$ as the cohomology of F on the final object of $\mathrm{PShv}(\mathcal{C})$. More canonically and precisely, we write $R\Gamma(\mathcal{C}, -)$ for the functor

$$D(\mathrm{Ab}(\mathrm{PShv}(\mathcal{C}))) \to D(\mathrm{Ab})$$

obtained by deriving the left exact functor $F \mapsto H^0(\mathcal{C}, F) := \lim_{X\in\mathcal{C}} F(X)$.

Example 4.2. Say $\mathcal{C}$ coincides with the poset $\mathbf{N} := \{0 \to 1 \to 2 \to ...\}$ of natural numbers. An abelian presheaf F on $\mathcal{C}$ then corresponds to a projective system $\{... \to F_2 \to F_1 \to F_0\}$ of abelian groups. It is an exercise to check that $R\Gamma(\mathcal{C}, F) \simeq R\lim_n F_n$ in this case.

For our purposes, we shall need the following recipe for computing cohomology.

Lemma 4.3. *Let $\mathcal{C}$ be a small category admitting finite non-empty products. Let F be an abelian presheaf on $\mathcal{C}$. Assume there exists a weakly final object $X \in \mathcal{C}$, i.e., $\mathrm{Hom}(Y, X) \neq \emptyset$ for every $Y \in \mathcal{C}$. Then $R\Gamma(\mathcal{C}, F)$ is computed by (the chain complex attached to) the cosimplicial abelian group*

$$F(X) \rightrightarrows F(X \times X) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} F(X \times X \times X) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots$$

obtained by applying F to the Cech nerve of X.

Proof sketch. This is a formal consequence of Cech theory once one observes that the hypothesis on X ensures that $\mathrm{Hom}(-, X)$ covers the final object in $\mathrm{PShv}(\mathcal{C})$. See [7, Tag 07JM]. ☐

5. Computing prismatic cohomology

We want to use Lemma 4.3 to compute prismatic cohomology. To do so, we need to find an object X as in Lemma 4.3, and also to ensure that $(R/A)_{\Delta}$ has finite non-empty coproducts (c.f. Remark 2.2). Both of these shall be a consequence of the following lemma, constructing "prismatic envelopes" of δ-pairs over (A, I).

Lemma 5.1. *Let (B, J) be a δ-pair over (A, I). Then there is a universal map $(B, J) \to (C, K)$ of δ-pairs over (A, I) where (C, K) is a prism (and thus $K = IC$). We sometimes write $C := B\{\frac{J}{I}\}^\wedge$ in this case.*

Proof sketch. Pick a generator $d \in I$. Consider the δ-ring B' obtained by formally adjoining $\frac{x}{d}$ for each $x \in J$ to B in the world of δ-A-algebras. Let B_1 be the derived (p, d)-completion (as modules) of the largest d-torsionfree quotient of B' in the world of δ-rings. If B_1 is d-torsionfree, then setting $C = B_1$ and $K = dB_1$ solves the problem. If not, one transfinitely iterates the operation passing to the maximal d-torsionfree quotient in δ-rings and derived (p, d)-completion (as modules) to arrive at the required ring C; the key point is that a countably filtered colimit of (p, d)-complete rings is already (p, d)-complete. ☐

The proof of the preceding lemma sheds almost no light on the nature of $B\{\frac{J}{I}\}^\wedge$. In fact, we do not expect this ring to be well-behaved in this generality. We shall later give a more meaningful and useful description (avoiding transfinite constructions) under extra hypotheses on J. For now, we simply note the following consequence.

Corollary 5.2. *The category $(R/A)_{\mathbb{\Delta}}$ admits finite non-empty coproducts.*

Proof. Say $(R \to B/IB \leftarrow B)$ and $(R \to C/IC \leftarrow C)$ are two objects of $(R/A)_{\mathbb{\Delta}}$. Consider the δ-ring $D_0 := B \otimes_A C$. Note that we have two natural maps $R \to D_0/ID_0$ defined by factoring through B/IB and C/IC respectively. We need to enlarge D_0 to ensure that these maps coincide. Let J denote the kernel of the natural map

$$D_0 \to B/IB \otimes_{A/IB} C/IC \to B/IB \otimes_R C/IC.$$

In other words, J is generated elements of the form $x \otimes 1 - 1 \otimes y$ where $x \in B$ and $y \in C$ have the property there exists some $z \in R$ with image x in B/IB and image y in C/IC. Applying Lemma 5.1 to the pair (D_0, J) gives a prism (D, ID) over (A, I). Moreover, by construction, we have a natural map $R \to D/ID$ defined via $R \to B/IB \to D_0/ID_0 \to D/ID$ or $R \to C/IC \to D_0/ID_0 \to D/ID$ (one checks that they coincide). The resulting object $(R \to D/ID \leftarrow D) \in (R/A)_{\mathbb{\Delta}}$ is then easily seen to be the desired coproduct. $\square$

We are now in a position to use Lemma 4.3 to obtain complexes computing prismatic cohomology.

Construction 5.3 (Cech-Alexander complexes for prismatic cohomology)**.** We begin by observing that there exists a free δ-A-algebra F_0 equipped with a surjection $F_0 \to R$. For example, with notation as in Example 2.7, we may simply take F_0 to be the free δ-A-algebra on the underlying set of $\widetilde{R}$; if one wishes to avoid making choices and have a strictly functorial construction, we may use the δ-A-algebra $W(R)$ instead of $\widetilde{R}$. Let $J \subset F_0$ be the kernel of $F_0 \to R$. Applying Lemma 5.1 to δ-pair (F_0, J) gives a prism (F, IF) over (A, I). Moreover, since $JF \subset IF$, we have an induced map $R \simeq F_0/J \to F/IF$. We can summarize this construction by the following commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d] & F := F_0\{\frac{J}{I}\}^\wedge \arrow[d] \\ A/I \arrow[r] & F_0/J \simeq R \arrow[r] & F/IF, \end{tikzcd}$$

where all arrows in the top row are δ-maps. In particular, we obtain an object

$$X := (R \to F/IF \leftarrow F) \in (R/A)_{\mathbb{\Delta}}.$$

We claim that X is a weakly initial object, i.e., there exists a map $X \to Y$ for any $Y \in (R/A)_{\mathbb{\Delta}}$. Using the universal property from Lemma 5.1, it is enough to show the following: for any object $(R \to B/IB \leftarrow B) \in (R/A)_{\mathbb{\Delta}}$, there exists a δ-map $F_0 \to B$ compatible with the map $R \to B/IB$. But this is clear as F_0 is a free δ-A-algebra: we simply send the generators in F_0 to arbitrary lifts in B of the images of these generators along $F_0 \to R \to B/IB$. Applying Lemma 4.3, we learn that $\mathbb{\Delta}_{R/A}$ is computed by a cosimplicial δ-A-algebra

$$F^0 \rightrightarrows F^1 \substack{\rightarrow\\[-1em] \rightarrow \\[-1em] \rightarrow} F^2 \substack{\rightarrow\\[-1em] \rightarrow \\[-1em] \rightarrow \\[-1em] \rightarrow} \cdots,$$

where F^n is obtained by applying $\mathcal{O}_{\mathbb{\Delta}}$ the $(n+1)$-fold self coproduct of the object X. In particular, $F^0 = F$, and each F^n is a d-torsionfree and (p, d)-complete δ-ring.

Let us explain how to use the Cech-Alexander complex to prove Lemma 3.6.

Lemma 5.4. *Fix an element $t \in R$. The class $\beta_d(\eta(t)) \in H^1(\overline{\mathbb{\Delta}}_{R/A})$ squares to 0 in $H^2(\overline{\mathbb{\Delta}}_{R/A})$.*

Note that the lemma reduces (by functoriality) to the case where $R = A/I\langle T\rangle$ and t is the generator T. In this case, it will follow from Theorem 3.8 that the group $H^2(\overline{\Delta}_{R/A})$ itself vanishes. However, since the lemma feeds into the formulation of Theorem 3.8, we must give a direct proof. We thank de Jong for the following proof.

Proof. We may assume $p = 2$ as the assertion is trivially true otherwise. By Construction 5.3, we may choose a cosimplicial d-torsionfree δ-A-algebra $F^\bullet$ calculating $\Delta_{R/A}$. Write $\partial(-)$ for the alternating sum of the face maps in $F^\bullet$, so $\partial(-)$ is the differential for the chain complex corresponding to $F^\bullet$. Note that $\partial(-)$ is A-linear and commutes with ϕ since the same holds true for all the face maps separately.

Choose some $T \in F^0$ lifting the image of $\eta(t)$ in F/IF. Write $U, V \in F^1$ and $X, Y, Z \in F^2$ for the images of T under the various structure maps $F^0 \to F^1$ and $F^0 \to F^2$ in the cosimplicial δ-ring $F^\bullet$, so $\partial(T) = U - V$ and $\partial(U) = X - Y + Z$. Note that there exists a (unique) $\alpha \in F^1$ with $U - V = d\alpha$: the images of U and V in F^1/dF^1 come from the same element (namely, t) in R. Moreover, since $U - V$ is a cycle (even a boundary) and F^2 is d-torsionfree, the element α is also a cycle. Unwinding the definition of the Bockstein differential shows that the image of α in $H^1(F^\bullet/dF^\bullet)$ coincides with $\beta_d(t)$. Our task is thus to check that $\alpha \cup \alpha = 0$ in $H^2(F^\bullet/dF^\bullet)$. Equivalently, it is enough to show that $(U - V) \cup (U - V)$ vanishes in $H^2(d^2F^\bullet/d^3F^\bullet)$. By definition, this cup product is computed by $(X - Y)(Y - Z) \in d^2F^2$, so we must show that $(X - Y)(Y - Z)$ defines the 0 element in $H^2(d^2F^\bullet/d^3F^\bullet)$. We will prove the more precise statement that $(X - Y)(Y - Z) \in d^2F^2$ is the boundary of $d^2\delta(\alpha) \in d^2F^1$. In particular, this cup product already vanishes in $H^2(d^2F^\bullet)$.

Since $p = 2$, one checks that we have a universal formula

$$\delta(a - b) = \delta(a) - \delta(b) + b(a - b).$$

Taking $a = U$ and $b = V$, and noting that $\delta(U) - \delta(V) = \partial(\delta(T))$, we get

$$\delta(U - V) = \epsilon + V(U - V)$$

where $\epsilon \in F^1$ is a boundary. Applying the differential gives

$$\partial(\delta(U - V)) = \partial(V(U - V)) = Y(X - Y) + Z(Y - Z) - Z(X - Z) = (X - Y)(Y - Z) \in d^2F^2.$$

It now remains to identify the left hand side with $\partial(d^2\delta(\alpha))$. For this, we apply $\delta(-)$ to $U - V = d\alpha$ to get

$$\delta(U - V) = d^2\delta(\alpha) + \phi(\alpha)\delta(d).$$

As the boundary maps in $F^\bullet$ are A-linear, it suffices to show that $\phi(\alpha) \in F^1$ is a cycle. But α is a cycle and ϕ commutes the differential in $F^\bullet$, so $\phi(\alpha)$ is indeed a cycle. $\square$

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