LECTURE IV: PERFECT PRISMS AND PERFECTOID RINGS

In this lecture, we study the commutative algebra properties of perfect prisms. These turn out to be equivalent to *perfectoid rings*, and most of the lecture studies algebraic properties of this class of rings. Some suitable references are [3, §8], [1, §3], [2, §3] and [4, §5].

1. Perfect prisms

Let us begin with an elementary lemma that shall be used repeatedly.

Lemma 1.1. *Let R be a perfect $\mathbf{F}_p$-algebra, and let $f \in R$. Then $R[f^\infty] = R[f^{1/p^m}]$ for any $m \geq 0$, and thus $R[f^\infty] = R[f^{1/p^\infty}] := \cap_m R[f^{1/p^m}]$. In particular, the derived f-completion of R coincides with the classical f-adic completion of R.*

Proof. The second part follows from the first part as the classical and derived f-completions coincide if the f^∞-torsion is bounded. For the first part, fix some $x \in R[f^\infty]$. Then $f^{p^n} x = 0$ for some $n \gg 0$. But then $f^{p^n} x^{p^{nm}} = 0$ for all $m \geq 0$. As R is perfect, taking roots shows that $f^{1/p^m} x = 0$. ☐

Using this lemma, we learn that perfect prisms are bounded in a rather strong sense.

Lemma 1.2. *Let (A, I) be a perfect prism. Then A is classically (p, I)-complete, $A/I[p^\infty] = A/I[p]$, and $A/p[I^\infty] = A/p[I]$. In particular, (A, I) is bounded.*

Proof. The $\mathbf{F}_p$-algebra A/p is perfect (by functoriality) and derived I-complete (as A is so). By Lemma 1.1, this algebra is also classically I-complete. By devissage, the same holds true for each A/p^n. As A is p-torsionfree, we have $A \simeq \lim_n A/p^n$ by derived p-completeness, so the limit A is also classically (p, I)-complete. In particular, we must have $A \simeq W(A/p)$ by the classification of perfect and p-adically complete δ-rings (Lemma II.3.2). The assertion $A/p[I^\infty] = A/p[I]$ follows from Lemma 1.1, while the assertion $A/I[p^\infty] = A/I[p]$ follows from the structure of distinguished elements in perfect δ-rings (Lemma III.1.10); a direct deduction of a stronger statement is also given in Lemma 2.6 below. The boundedness of (A, I) is then clear. ☐

There is a perfection operation for prisms.

Lemma 1.3 (Perfection of prisms)**.** *For any prism (A, I), there is an initial map $(A, I) \to (A_\infty, IA_\infty)$ to a perfect prism (A_∞, IA_∞).*

Proof. Let $A'_\infty := \varinjlim_\phi A$ be the "perfection" of A in the category of δ-rings. Set A_∞ to be the derived (p, I)-completion of A. We claim that the resulting map $(A, I) \to (A_\infty, IA_\infty)$ of δ-pairs satisfies the conclusion of the lemma. In fact, the universality is clear once we know that (A_∞, IA_∞) is a perfect prism. Unwinding definitions, it is enough to check that A_∞ is a perfect δ-ring, and that $IA_\infty = (d)$ for a distinguished element $d \in A_\infty$ which is a nonzerodivisor.

Let us first show that A_∞ is a perfect and classically p-adically complete δ-ring. Note that A_∞ is a δ-ring thanks to preservation of δ-structures under derived (p, I)-completion (Exercise III.2.7). The associated Frobenius endomorphism is an automorphism as the same was true for A'_∞ (and because derived (p, I)-completion and derived $(p, \phi(I))$-completion coincide). Thus, A_∞ is a perfect and derived (p, I)-complete δ-ring. As any perfect δ-ring is p-torsionfree (Lemma II.3.3), it follows that A_∞ is p-torsionfree and thus also classically p-adically complete. By the classification of perfect and classically p-complete δ-rings (Proposition II.3.2), we learn that A/p is a perfect $\mathbf{F}_p$-algebra and $A \simeq W(A/p)$ as δ-rings.

It remains to show that $IA_\infty = (d)$ for a distinguished element $d \in A_\infty$ which is a nonzerodivisor. By Lemma III.3.5, the ideal $\phi(I)A_\infty$ is principal ideal generated by a distinguished element. As A_∞ is perfect, the map ϕ is an automorphism, so $IA_\infty = (d)$ for a distinguished element d. As we have already shown that A_∞ perfect and classically p-adically complete, the classification of distinguished elements in perfect δ-rings (Lemma III.1.10) shows that d is a nonzerodivisor. $\square$

2. Perfectoid rings

Let us use the language of prisms to give a slightly non-standard definition of perfectoid rings.

Definition 2.1. A commutative ring R is called *perfectoid* if it has the form A/I for a perfect prism (A, I). The category of perfectoid rings is the full subcategory of all commutative rings spanned by perfectoid rings.

All our previously noted examples of perfect prisms give perfectoid rings; in particular, any perfect $\mathbf{F}_p$-algebra is a perfectoid ring corresponding to a crystalline prism. As any perfect prism is bounded (Lemma 1.2), all perfectoid rings are classically p-adically complete (Lemma II.2.4).

Remark 2.2. The definition introduced above will be shown to coincide with the one from [2]. However, it is sometimes called *integral perfectoid* in the literature to emphasize its integral nature, and to contrast with the notion of a perfectoid Tate ring introduced in [4].

Our first result is that passing from a perfect prism to the associated perfectoid ring loses no information.

Theorem 2.3 (Perfect prisms $\cong$ perfectoid rings)**.** *The construction $(A, I) \mapsto A/I$ defines an equivalence of categories between perfect prisms and perfectoid rings.*

Proof. Let (A, I) be a perfect prism with associated perfectoid ring R. We shall explain how to recover (A, I) from R; the rest of the verifications shall be left to the reader.

To recover A, we define $R^\flat := \varprojlim_\phi R/p$ to be the inverse limit perfection of R/p. We shall show that $A \simeq W(R^\flat)$. As both sides are perfect and p-adically complete δ-rings, it is enough to show that $A/p \simeq R^\flat$ (Proposition II.3.2). Note that $R/p = A/(p, d)$. As A/p is perfect, it is then easy to see that $R^\flat$ identifies with the classical d-adic completion of A/p: the n-fold Frobenius map $A/(p, d) \to A/(p, d)$ identifies with the canonical map $A/(p, d^{p^n}) \to A/(p, d)$ compatibly in n, so the inverse limit $\varprojlim_\phi R/p$ identifies with the inverse limit $\varprojlim_n A/(p, d^{p^n})$. It is thus enough to show that A/p is classically d-complete, which follows from Lemma 1.1.

Next, we explain how to recover $I \subset A$ from $A := W(R^\flat)$ and R. By construction, we have a surjective map $R^\flat \to R/p$. By deformation theory (i.e., vanishing of the cotangent complex $L_{(A/p)/\mathbf{F}_p}$), this lifts uniquely to a map $A := W(R^\flat) \to R$. This map is surjective by derived Nakayama. By uniqueness, this map $A \to R$ must coincide with the original map $A \to A/I =: R$ as they agree modulo p. In particular, we can recover I as the kernel of $A := W(R^\flat) \to R$. $\square$

In the rest of this section, we study some algebraic properties of perfectoid rings. The proofs ultimately rest on rather surprisingly nice features of perfect $\mathbf{F}_p$-algebras. We record some such features in the next exercise, and we strongly encourage the reader to try this exercise.

Exercise 2.4 (Radical ideals in perfect $\mathbf{F}_p$-algebras)**.** Let R be a $\mathbf{F}_p$-algebra. For any ideal $I \subset R$, write $I^{[p^e]} := \{x^{p^e} \mid x \in I\}$ for any $e \in \mathbf{Z}_{\geq 0}$. When R is perfect, this makes sense for all $e \in \mathbf{Z}$. Assume now that R is perfect.

(1) Fix an ideal $I \subset R$. Show that the following are equivalent:
   (a) I is radical.

(b) $I^{[p]} = I$.

(c) I has the form $(\{f_i^{1/p^\infty}\}_{i \in S}) := \cup_n (\{f_i^{1/p^n}\}_{i \in S})$, where S is some set.

(2) (Aberbach-Hochster) Let $f_1, ..., f_r \in R$, and consider the ideal $I = \sqrt{(f_1, .., f_r)} \subset R$. Show that R/I has flat dimension $\leq r$ as an R-module. (Hint for $r = 1$: for $I = \sqrt{fR}$, show that $I = (f^{1/p^\infty})$ and that the natural map

$$\varinjlim(R \xrightarrow{f^{1-1/p}} R \xrightarrow{f^{1/p-1/p^2}} R \xrightarrow{f^{1/p^2-1/p^3}} R \to ....) \to I$$

is an isomorphism.)

(3) If $B \leftarrow A \to C$ are maps of perfect rings, then show that $\mathrm{Tor}_i^A(B, C) = 0$ for $i > 0$. (Hint: reduce to the case where $A \to B = A/I$ is a surjection with $I = (f^{1/p^\infty})$, and then use (2).)

The following notions associated to a perfectoid ring historically predate the above definition.

Definition 2.5. Let R be a perfectoid ring, and write (A, I) for the corresponding perfect prism.

(1) The *tilt* $R^\flat$ *of* R is the perfect $\mathbf{F}_p$-algebra defined as $\varprojlim_\phi R/p$. By the proof of Theorem 2.3, this also coincides with A/p. (Note that the definition of the tilt makes sense for any commutative ring.)

(2) Write $A_{\mathrm{inf}}(R) := A$ and $\theta_R : A_{\mathrm{inf}}(R) \to R$ for the resulting map. When R is clear from the context, we also write A_{inf} and θ instead of $A_{\mathrm{inf}}(R)$ and θ_R. (Note that the proof of Theorem 2.3 shows that $A_{\mathrm{inf}}(R) \simeq W(R^\flat)$.)

(3) Write $\overline{R} = R/\sqrt{pR}$; this is a perfect ring of characteristic p (see proof of Lemma 2.6 (2)), and we sometimes call it the *special fibre* of R.

In fact, the notions in (1) and (2) extend to arbitrary p-adically complete rings; this is essentially clear by the proof of Theorem 2.3, and elaborated on in Exercise 2.9.

Perfectoid rings will be shown to support a good theory of "almost mathematics". For this, we need the following lemma describing the structure of the ideal $\sqrt{pR}$ for a perfectoid ring R.

Lemma 2.6. *Let R be a perfectoid ring.*

(1) The Frobenius on R/p is surjective.

(2) There exists an element $\varpi \in R$ that admits a compatible system ϖ^{1/p^n} of p-power roots such that $\varpi = pu$ for a unit u and such that the kernel of the Frobenius map $R/p \to R/p$ is generated by $\varpi^{1/p}$.

(3) The ideal $\sqrt{pR}$ is flat, an increasing union of principal ideals, and satisfies $(\sqrt{pR})^2 = \sqrt{pR}$. In fact, with notation as in (2), we have $\sqrt{pR} = \cup_n(\varpi^{1/p^n})$.

(4) We have $R[p] = R[\sqrt{pR}]$.

Proof. Write $A = A_{\mathrm{inf}}(R) = W(R^\flat)$, so $R = A/(d)$ for a distinguished element $d \in A$. Then $R/p = A/(p, d)$, so (1) is clear as $A/p = R^\flat$ is perfect.

For (2), note that we can write $d = \sum_{i \geq 0} [a_i] p^i$ with a_1 being a unit. Combining terms, we can write $d = [a_0] - pu$ for a unit $u \in A$. Writing $\varpi \in R$ for the image of $[a_0]$ then shows that $\varpi = pu$ in R for a unit u. Setting ϖ^{1/p^n} to be the image of $[a_0^{1/p^n}]$ gives the desired compatible system of p-power roots. The Frobenius map $R/p \to R/p$ identifies with the Frobenius map $A/(p, [a_0]) \to A/(p, [a_0])$, which has kernel generated by $[a_0^{1/p}]$ since A/p is perfect. This translates to the statement that $\varpi^{1/p}$ generates the kernel of the Frobenius on R/p.

Choose ϖ^{1/p^n} as in (2). Let us first show that $\sqrt{pR} = \cup_n \varpi^{1/p^n} R$; note that this immediately implies that $\sqrt{pR} = (\sqrt{pR})^2$. As the containment $\cup_n \varpi^{1/p^n} R \subset \sqrt{pR}$ is clear, it suffices to show that $R/(\cup_n \varpi^{1/p^n} R)$ is reduced. But this ring as the form $S/x^{1/p^\infty}S$ for a perfect $\mathbf{F}_p$-algebra S; any such quotient is perfect and thus reduced.

To finish (3), we check flatness of $\sqrt{pR}$. Write $\overline{R} := R/\sqrt{pR}$, so $\overline{R}$ is a perfect $\mathbf{F}_p$-algebra. It suffices to show that for any bounded $M \in D^{\geq 0}(R)$, we have $F(M) := M \otimes^L_R R/\sqrt{pR} \in D^{\geq -1}$. By consideration of the triangle $M \otimes^L_{\mathbf{Z}_p} \mathbf{Q}_p/\mathbf{Z}_p[-1] \to M \to M[\frac{1}{p}]$ and the fact that $\overline{R}[\frac{1}{p}] = 0$, it suffices to show that $F(M) \in D^{\geq -1}(R)$ for any bounded $M \in D^{\geq 0}(R)$ whose cohomology groups are p^∞-torsion. In fact, by filtering M in terms of its cohomology modules, we may even assume that M is a p^∞-torsion R-module. Using the compatibility of tensor products with direct limits, we reduce to the case where M is an R/p^n-module for some $n \geq 1$. By filtering further, we may assume $n = 1$, i.e., that M is an R/p-module. Now consider the commutative square

$$\begin{tikzcd} W(R^\flat) = A_{\mathrm{inf}}(R) \arrow[r] \arrow[d] & R \arrow[d] \\ W(\overline{R}) = A_{\mathrm{inf}}(\overline{R}) \arrow[r] & \overline{R}. \end{tikzcd}$$

As distinguished elements are nonzerodivisors in perfect p-adically complete δ-rings (Lemma III.1.10), this square is a Tor-independent pushout square. We therefore have $M \otimes^L_R \overline{R} \simeq M \otimes^L_{W(R^\flat)} W(\overline{R})$. As p is a nonzero divisor on both $W(R^\flat)$ and $W(\overline{R})$ and $pM = 0$, we also have $M \otimes^L_{W(R^\flat)} W(\overline{R}) \simeq M \otimes^L_{R^\flat} \overline{R}$. Now $R^\flat \to \overline{R}$ is a quotient by the ideal of the form (f^{1/p^∞}), so the claim follows from Exercise 2.4.

For (4), choose $a_0 \in R^\flat$ and the distinguished element $d = \sum_{i \geq 0} [a_i] p^i$ as above. As discussed above, the ideal $\sqrt{pR}$ is generated (by the images of) $[a_0^{1/p^n}]$ for $n \geq 0$. It is thus enough to show that the A-module $R[p]$ is annihilated by $[a_0^{1/p^n}]$ for $n \geq 0$. As both p and d are nonzerodivisors in the ring A, the torsion exchange Lemma 2.7 shows that $A/p[d] \simeq A/d[p]$ as A-modules. It is thus enough to show that the A-module $A/p[d]$ is annihilated by $[a_0^{1/p^n}]$ for $n \geq 0$. But $A/p = R^\flat$ and $d = a_0$ in A/p, so the claim follows from Lemma 1.1. $\square$

The following lemma was used above and will be useful later.

Lemma 2.7 (Torsion exchange lemma)**.** *Let A be a commutative ring. Let $x, y \in A$ be a nonzero-divisors. Then $A/x[y] \simeq A/y[x]$ as A-modules.*

Proof. Both A-modules under consideration identify with H_1 of $\mathrm{Kos}(A; x, y) \simeq \mathrm{Kos}(A; y, x)$. $\square$

A perfectoid ring R is related to its tilt $R^\flat$ by the "correspondence" $R \to R/p \leftarrow R^\flat$ of rings. It turns out that if one ignores additive structures and only keeps track of multiplicative structures, we can lift this correspondence to a map $R^\flat \to R$.

Definition 2.8 (The $\sharp$-map)**.** Let R be a perfectoid ring. The $\sharp$-map is the multiplicative map $R^\flat \to R$, denoted $x \mapsto x^\sharp$, and defined as the composition $R^\flat \xrightarrow{[\cdot]} W(R^\flat) \simeq A_{\mathrm{inf}}(R) \to R$.

Unwinding definitions shows that the composition $R^\flat \xrightarrow{\sharp} R \to R/p$ coincides with the projection $R^\flat \simeq \varprojlim_\phi R/p \to R/p$ to the last component. There is a slightly more useful description of this map recorded next.

Exercise 2.9 (Fontaine's θ-map in general)**.** Let R be any commutative ring, and write $R^\flat := \varprojlim_\phi R/p$ for *the tilt* of R. Consider the inverse limit monoid $\varprojlim_{x \mapsto x^p} R$ parametrizing sequences $\{x_n \in R\}$ with $x_{n+1}^p = x_n$. Assume R is p-adically complete for the exercises that follow.

(1) Show that the obvious map $\varprojlim_{x\mapsto x^p} R \to R^\flat$ given by reduction mod p on the terms is a multiplicative bijection.
(2) Composing the inverse of the bijection in (1) with projection onto the first component of $\varprojlim_{x\mapsto x^p} R$, we obtain a multiplicative map $R^\flat \to R$. Show that the image of this map is exactly those elements of R that admit a compatible system of p-power roots. For R perfectoid, show that this agrees with the $\sharp$-map.
(3) Using deformation theory, show that the canonical map $R^\flat \to R/p$ (of rings) lifts to a map $W(R^\flat) \to R$ of rings. Thus, if one defines $A_{\mathrm{inf}}(R) := W(R^\flat)$, one can make sense of the map $\theta_R : A_{\mathrm{inf}}(R) \to R$ from Definition 2.5 for arbitrary p-adically complete rings.

Using the above map, one can give an intrinsic definition of a perfectoid ring.

Proposition 2.10 (Characterizing perfectoid rings). *A commutative ring R is perfectoid if and only if the following conditions hold:*

(1) R is classically p-adically complete.
(2) The Frobenius $R/p \to R/p$ is surjective.
(3) The kernel of $\theta_R : A_{\mathrm{inf}}(R) \to R$ (which exists because R is p-adically complete) is principal.
(4) There exists some $\varpi \in R$ such that $\varpi^p = pu$ for a unit $u \in R$.

If R is p-torsionfree, condition (3) can be replaced by the following condition:

(3') If $x \in R[1/p]$ with $x^p \in R$, then $x \in R$.

Proof. We have already seen that a perfectoid rings satisfies (1) through (4). Conversely, assume R is a commutative ring satisfying (1) through (4). The map θ_R is surjective: its mod p reduction is the projection $R^\flat = \lim_\varphi R/p \to R/p$, which is surjective by (2). Let $d \in A_{\mathrm{inf}}(R)$ denote a generator of the kernel of θ_R. It suffices to show d is distinguished. Note that $A_{\mathrm{inf}}(R)$ is derived (p,d)-complete: derived p-completeness is clear, and derived d-completeness can be checked modulo p where it follows from the derived f-completeness of $R^\flat$ for any f in the kernel of $R^\flat \to R/p$. Fix $\varpi, u \in R$ as in (4), and lift them $x, v \in A_{\mathrm{inf}}(R)$. As $A_{\mathrm{inf}}(R)$ is derived d-complete, $v \in A_{\mathrm{inf}}(R)^*$ while $x \in \mathrm{Rad}(A_{\mathrm{inf}}(R))$. The element $g := pv - x^p \in A_{\mathrm{inf}}(R)$ then lies in $\ker(\theta_R)$, so $g = wd$ for some $w \in A_{\mathrm{inf}}(R)$. By the irreducibility of distinguished elements (Lemma II.1.7), it will suffice to show that g is distinguished. But this can be seen immediately using, for example, Teichmuller expansions: the coefficient of p in any p-th power in $W(R^\flat)$ is 0, so the coefficient of p in $pv - x^p$ coincides with the unit $[\overline{v}]$, where $\overline{v} \in R^\flat$ is the image of the unit $v \in W(R^\flat)$. Alternately, the canonical map $\eta : W(R^\flat) \to W(R^\flat/\mathrm{Rad}(R^\flat))$ is a surjective map of δ-rings that carries g to the distinguished element pv, which implies that g must be distinguished as $\ker(\eta) \subset \mathrm{Rad}(W(R^\flat))$.

Now assume R is a p-torsionfree perfectoid ring. Let us show R satisfies $(3')$. Fix some $x \in R[1/p]$ with $x^p \in R$. By Lemma 2.6, we may choose some $\varpi \in R$ such that $\varpi^p = pu$ for a unit u and such that the Frobenius map $R/\varpi \to R/\varpi^p$ is bijective. Choose the minimal integer $n \geq 0$ such that $y := \varpi^n x \in R$. We must show $n = 0$. Assume that $n > 0$ towards contradiction. Since $x^p \in R$, we have $(\varpi^n x)^p = \varpi^{np} x^p \in \varpi^{np} R \subset \varpi^p R$. Using the injectivity of Frobenius $R/\varpi \to R/\varpi^p$, we get that $\varpi^n x \in \varpi R$, whence $x \in \varpi^{n-1} R$ as R is ϖ-torsionfree. But this conradicts the minimality to n, so we must have had $n = 0$, as wanted.

Conversely, assume R is a p-torsionfree ring satisfying (1), (2), (4) and (3'). We will show R satisfies (3) and is thus perfectoid. Let us first show that the surjective Frobenius map $\varphi : R/p \to R/p$ has kernel generated by any element $\varpi \in R$ as in (4). Given $x \in R$ with $x^p \in pR$, we must show that $x \in \varpi R$. As $p = \varpi^p u$, if $x^p \in pR$, we can write $x^p = \varpi^p y$ for some $y \in R$. But then $x/\varpi \in R[1/p]$ has its p-th power in R, so $x/\varpi \in R$ by (3'), whence $x \in \varpi R$, as wanted. In particular, $\varphi : R/p \to R/p$ factors as

$$\varphi : R/p \xrightarrow{can} R/\varpi \xrightarrow{\tilde{\varphi}} R/p,$$

where the first map is the canonical quotient, while the second map is induced by the p-power map on R and is an isomorphism. As the Frobenius map $R/p \to R/p$ is surjective, the image $\overline{\varpi} \in R/p$ of $\varpi \in R$ admits a compatible system $\overline{\varpi}^{1/p^n} \in R/p$ of p-power roots that we fix once and for all. Using the preceding factorization of φ, it is easy to see by induction that $\ker(\phi^n) = (\overline{\varpi}^{1/p^{n-1}})$: we already checked it for $n = 1$, and the general case follows as $\ker(\varphi^n) = \varphi^{-1}(\ker(\varphi^{n-1})) = \varphi^{-1}((\varpi^{1/p^{n-2}})) = (\varpi^{1/p^{n-1}})$, where the last equality is seen by the factorization of φ above. Passing to limits, it follows that the kernel of the projection $\overline{\theta_R} : R^\flat \to R/p$ is generated by the element $\varpi^\flat$ determined by the compatible system $\{\overline{\varpi}^{1/p^n}\} \in \lim_\varphi R/p$. As both $W(R^\flat)$ and R are p-torsionfree and p-adically complete, the kernel of the map $\theta_R : A_{\mathrm{inf}}(R) = W(R^\flat) \to R$ is then generated by any element in this kernel that lifts $\varpi^\flat$. In particular, this kernel is principal, so we R satisfies (3) and is thus perfectoid. □

The category of perfectoid rings is closed under pushouts in the category of all derived p-complete rings, even in a derived sense.

Proposition 2.11 (Pushouts of perfectoids rings)**.** *Let $C \leftarrow A \to B$ be maps of perfectoid rings. Then the derived p-completion of $B \otimes_A^L C$ has cohomology just in degree 0, where it is a perfectoid ring. In short, $B \widehat{\otimes}_A^L C$ is a perfectoid ring.*

Proof. Consider the diagram

$$\begin{tikzcd} A^\flat \arrow[r] \arrow[d] & B^\flat \arrow[d] \\ C^\flat \arrow[r] & R \end{tikzcd}$$

where $R = C^\flat \otimes_{A^\flat} B^\flat$. It is clear that R is a perfect ring: the Frobenius on R is defined by functoriality from the Frobenii on the rest of the diagram. By Exercise 2.4, we also have $R \simeq C^\flat \otimes^L_{A^\flat} B^\flat$, i.e., derived pushouts of perfect rings are automatically discrete. Applying the Witt vectors then gives a diagram

$$\begin{tikzcd} W(A^\flat) \arrow[r] \arrow[d] & W(B^\flat) \arrow[d] \\ W(C^\flat) \arrow[r] & W(R) \end{tikzcd}$$

which expresses $W(R)$ as the derived pushout $W(A^\flat) \widehat{\otimes}^L_{W(A^\flat)} W(B^\flat)$ of the rest of the picture (here $\widehat{\otimes}^L$ refers to the derived p-completion of the derived tensor product): by derived Nakayama, this assertion can be checked modulo p where it follows from the previous one. Fix a distinguished element $d \in W(A^\flat)$ such that $W(A^\flat)/(d) \simeq A$. Applying derived p-complete base change along $W(A^\flat) \to A$ then gives a derived pushout diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D := W(R)/(d), \end{tikzcd}$$

where we implicitly use the following: distinguished elements in perfect p-adically complete δ-rings are nonzerodivisors (Lemma III.1.10), and finitely presented quotients of derived p-complete rings are derived p-complete. Summarizing, we have shown that $B\widehat{\otimes}^L_A C$ is simply the perfectoid ring D, as wanted. $\square$

Exercise 2.12. We study limits and colimits in perfectoid rings.

(1) Show that the category of perfect rings is closed under arbitrary limits and colimits in the category of all rings.
(2) Show that the category perfectoid rings is closed under arbitrary colimits and products in the category of all derived p-complete rings.
(3) Give an example to show that the category of perfectoid rings is not closed under equalizers in the category of all rings. (Hint: use Tate's theorem that $(\mathbf{C}_p)^{\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)} = \mathbf{Q}_p$, where $\mathbf{C}_p$ is the completion of the algebraic closure of $\mathbf{Q}_p$.)

3. A structure theorem for perfectoid rings

All perfectoid rings encountered so far were either p-torsionfree or entirely p-torsion. However, this does not always need to be the case: as the category of perfectoid rings has products, one can obtain "mixed" examples by taking products. One obtains slightly more complicated examples by passing to fibre products. We shall soon show that this is essentially all we can do. For this, we need the following easy fact about perfect $\mathbf{F}_p$-algebras.

Lemma 3.1. *Let R be a perfect $\mathbf{F}_p$-algebra. Let $I \subset R$ be a radical ideal, and let $J = R[I] \subset R$. Then J and $I + J$ are both radical, and the square*

$$\begin{array}{ccc} R & \longrightarrow & R/I \\ \downarrow & & \downarrow \\ R/J & \longrightarrow & R/(I+J) \end{array}$$

is both a homotopy[1] fibre square and a pushout square of commutative rings.

Proof. Let us first show that J is radical. If $x \in R$ with $x^p \in J$, then $x^p I = 0$. As R is perfect, this means $xI^{[1/p]} = 0$. As I is radical, we have $I = I^{[1/p]}$, so $xI = 0$, whence $x \in J$, proving that J is radical. To show that $I + J$ is radical, we simply note that $I + J$ is the kernel of $R \to R/I \otimes_R R/J$, and that the target of the preceding map is perfect as colimits of perfect rings are perfect.

It is trivial to see that the above square is a pushout square of commutative rings. To show it is also a pullback square, we must show that $I \cap J = 0$. Fix $x \in I \cap J$. Since $x \in J$, we have $xI = 0$. But $x \in I$, so we get that $x^2 = 0$ whence $x^p = 0$ and thus $x = 0$ as R is perfect. (The "homotopy" part of the assertion is automatic as the vertical maps are surjective.) $\square$

We now arrive at the promised structure theorem for perfectoid rings.

[1] A square of abelian groups is called a homotopy fibre square if the corresponding square in the derived category is a homotopy fibre square, i.e., the cones of the two horizontal maps (or equivalently the two vertical maps) are isomorphic. In general, this is stronger than requiring that the square of abelian groups merely be a fibre square in the category of abelian groups. However, it is an exercise to see that the two notions coincide if the vertical maps (or if the horizontal maps) are surjective. The reason to work with the derived notion is that it behaves better under certain operations, see proof of Proposition 3.2.

Proposition 3.2. *Let R be a perfectoid ring. Let $\overline{R} = R/\sqrt{pR}$, $S = R/R[\sqrt{pR}]$, and $\overline{S} = S/\sqrt{pS}$. Then $\overline{R}$, S, and $\overline{S}$ are perfectoid, and the resulting commutative square of rings*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ \overline{R} \arrow[r] & \overline{S} \end{tikzcd}$$

is both a fibre square and a pushout square. Moreover, we have the following:

(1) S is p-torsionfree.

(2) $\sqrt{pR}$ maps isomorphically onto $\sqrt{pS}$ (and is thus p-torsionfree).

(3) $R[\sqrt{pR}]$ maps isomorphically to $\ker(\overline{R} \to \overline{S})$, and thus $x \mapsto x^p$ is bijective on $R[\sqrt{pR}]$.

Crucially, the above lemma describes R as a (homotopy) fibre product of a diagram involving perfectoid rings that are either perfect of characteristic p or p-torsionfree.

Proof. We have already seen in Lemma 2.6 that $R[\sqrt{pR}] = R[p^\infty]$. In particular, it is immediate that S is p-torsionfree, giving (1). Once we know that the above square is a fibre square of perfectoid rings, consideration of the kernels of the horizontal and vertical maps will show (2) and (3). Moreover, either (2) or (3) trivially implies the square is a pushout square. So it remains to show that the square above is a pullback square.

Write $d = [a_0] - pu$ for a distinguished element of $A = A_{\mathrm{inf}}(R)$ such that $R = A/(d)$. Consider the perfect ring $R^\flat$ with ideals $I = (a_0^{1/p^\infty})$ and $J = R^\flat[I]$. The square

$$\begin{tikzcd} W(R^\flat) \arrow[r] \arrow[d] & W(R^\flat/J) \arrow[d] \\ W(R^\flat/I) \arrow[r] & W(R^\flat/(I+J)) \end{tikzcd}$$

is a homotopy fibre square by Lemma 3.1 and an easy p-adic devissage: all 4 vertices are p-torsionfree and p-adically complete, and the square gives a fibre square mod p^n for all n by reduction to the $n = 1$ case. Applying $- \otimes^L_{W(R^\flat)} R$, and noting that d is a nonzerodivisor on all 4 rings above (as they are perfect and p-adically complete), we get a homotopy fibre square

$$\begin{tikzcd} W(R^\flat)/(d) \simeq R \arrow[r] \arrow[d] & W(R^\flat/J)/(d) \arrow[d] \\ W(R^\flat/I)/(d) \arrow[r] & W(R^\flat/(I+J))/(d). \end{tikzcd}$$

Note that all 4 vertices of this diagram are perfectoid rings and that all the maps are surjective. We shall identify this square with the square in the statement of the lemma to finish the proof.

Let us first verify that the bottom left vertex $W(R^\flat/I)/(d)$ is simply $\overline{R}$. Note that $(d) = (p)$ inside $W(R^\flat/I)$ since $a_0 \in I$, so $W(R^\flat/I)/(d) = R^\flat/I \simeq R^\flat/(a_0^{1/p^\infty}) \simeq R/\sqrt{pR} = \overline{R}$, as wanted.

Next, we study the top right vertex $S' := W(R^\flat/J)/(d)$. We claim that this perfectoid ring is p-torsionfree. For this, applying Lemma 2.7 applied to $x = p$ and $y = d$ in $W(R^\flat/J)$ gives $(R^\flat/J)[d] = S'[p]$. But $(R^\flat/J)[d] = 0$ since $d = a_0$ is a nonzerodivisor in $R^\flat/J$: if $a_0 b = 0$ in this ring, then we can lift this to an equality of the form $a_0\tilde{b} = c$ in $R^\flat$ with $c \in R^\flat[a_0^{1/p^\infty}]$, whence $a_0^2\tilde{b} = 0$, so $\tilde{b} \in R^\flat[a_0^2]$, which equals $R^\flat[a_0^{1/p^\infty}]$ by Lemma 1.1. It follows that we must also have $S'[p] = 0$, so S' is p-torsionfree. Thus, the surjection $R \to S'$ in the top row factors uniquely as $R \to R/R[p^\infty] = S \to S'$. The reasoning used in the previous paragraph also shows that the bottom right vertex is $W(R^\flat/(I+J))/(d)$ is $\overline{S'}$.

To prove that the second map in the composition $R \to R/R[p^\infty] = S \to S'$ is an isomorphism, it remains to check that the kernel K of $R \to S'$ is contained in $R[p^\infty]$, i.e., is p-power torsion. But the fact that the above square is a fibre square ensures that K embeds into the bottom left vertex $\overline{R}$. But $\overline{R}$ has characteristic p, so K is indeed p-torsion, as wanted. □

Corollary 3.3. *Perfectoid rings are reduced.*

Proof. Let R be a perfectoid ring. By Proposition 3.2, we may assume that R is either p-torsionfree or perfect of characteristic p. As perfect rings are clearly reduced, we may assume R is p-torsionfree. Choose $\varpi \in R$ such that $\varpi^p = pu$ for a unit $u \in R$. Fix $x \in R$ such that $x^p = 0$. We shall inductively show that $x \in \varpi^n R$ for all n, whence $x = 0$ by ϖ-adic separatedness. Assume that $x \in \varpi^n R$, so $x = \varpi^n y$ for some $y \in R$. Then $\varpi^{np} y^p = 0$ and thus $y^p = 0$ as well since R is ϖ-torsionfree. As the kernel of the Frobenius $R/p \to R/p$ is generated by ϖ, we must have $y \in \varpi R$, whence $x \in \varpi^{n+1} R$ as wanted. □

References

[1] B. Bhatt, S. Iyengar, S. Ma, *Regular rings and perfect(oid) rings*
[2] B. Bhatt, M. Morrow, P. Scholze, *Integral p-adic Hodge theory*
[3] E. Lau, *Dieudonne theory over semiperfect and perfectoid rings*
[4] P. Scholze, *Perfectoid spaces*