LECTURE III: DISTINGUISHED ELEMENTS AND PRISMS

In this lecture, we introduce prisms. Modulo various completeness hypotheses, a prism consists of a δ-ring A together with a Cartier divisor $Z \subset \mathrm{Spec}(A)$ such that Z and $\phi^{-1}(Z)$ meet only in characteristic p.

We fix some notation for the rest of the lectures. Fix a prime p. For any commutative ring A, write $\mathrm{Rad}(A)$ for the Jacobson radical of A. Unless otherwise specified, assume all rings that appear are p-local, i.e., $p \in \mathrm{Rad}(A)$; this condition is ensured if A is p-adically complete. To ensure that the condition $p \in \mathrm{Rad}(A)$ survives various ring theoretic operations, we shall repeatedly use the following fact: if any A is a δ-ring and $Z \subset \mathrm{Spec}(A/p)$ is a closed subset, then the localization of A along Z has a unique δ-structure; this follows by applying Lemma II.2.7 to the multiplicative set $S \subset A$ consisting of all elements of A that are invertible on Z.

1. Distinguished elements

Definition 1.1. Let R be a δ-ring. An element $d \in R$ is called *distinguished* if $\delta(d)$ is a unit.

Note that any morphism of δ-rings preserves distinguished elements. Moreover, as ϕ commutes with δ and because our rings are p-local, applying ϕ preserves and detects distinguished elements, i.e., d is distinguished if and only if $\phi(d)$ is distinguished. Note that a distinguished element can never have rank 1 in a non-zero δ-ring.

Example 1.2. The following examples of distinguished elements d in a δ-ring A are crucial for cohomological purposes.

(1) (*Crystalline cohomology*) Take $A = \mathbf{Z}_{(p)}$ with $d = p$. Indeed, $\delta(p) = 1 - p^{p-1} \in \mathbf{Z}_{(p)}^*$. More generally, the image of p in any p-local δ-ring is distinguished. More generally, for any δ-ring A with $p \in \mathrm{Rad}(A)$, the image of $p \in A$ is distinguished. In particular, as there exist δ-rings with p-torsion, distinguished elements need not be nonzerodivisors.

(2) (*q-de Rham cohomology*) Take $A = \mathbf{Z}_p[\![q-1]\!]$, $d = [p]_q := \frac{q^p-1}{q-1} = \sum_{i=0}^{p-1} q^i \in A$, with δ-structure determined by $\phi(q) = q^p$. The distinguishedness of d can be seen directly. Alternately, as A is $(q-1)$-adically complete, an element of A is a unit if and only if its image under the δ-map $A \to A/(q-1) \simeq \mathbf{Z}_p$ (where the isomorphism is $q \mapsto 1$) is a unit. Now $[p]_q \equiv p \mod (q-1)$, so $\delta([p]_q) \equiv \delta(p) \mod (q-1)$, so the claim follows from (1).

(3) (*Breuil-Kisin cohomology*) Fix a discretely valued extension $K/\mathbf{Q}_p$ with uniformizer π. Let $W \subset \mathcal{O}_K$ be the maximal unramified subring. Take $A = W[\![u]\!]$ with δ-structure determined by the canonical one on W and satisfying $\phi(u) = u^p$. There is a W-equivariant surjection $A \to \mathcal{O}_K$ determined by $u \mapsto \pi$. Any generator $d \in A$ of the kernel of this map is distinguished; this can be seen like (2) via specialization along the δ-map $W[\![u]\!] \xrightarrow{u \mapsto 0} W$.

(4) (*A_{inf}-cohomology*) Let A be the $(p, q-1)$-adic completion of $\mathbf{Z}_p[q^{\frac{1}{p^\infty}}]$. Then A is p-torsionfree, and setting $\phi(q) = q^p$ gives a Frobenius lift; note that this map is an automorphism. It follows from Example (2) above that $d = [p]_q \in A$ is a distinguished element. As ϕ is an automorphism, each $\phi^n(d)$ is a distinguished element for any $n \in \mathbf{Z}$.

Remark 1.3. Let us explain the relevance of the names chosen above. In each example above, if one sets $I = (d) \subset A$, then the pair (A, I) has the following feature: the de Rham cohomology of a formally smooth A/I-scheme X admits a certain canonical deformation to A given by the cohomology theory named above. For example, in the context of example (4), this deformation is given by the A_{inf}-cohomology theory from [1]. By the end of this course, we aim to give a uniform construction of such deformations.

Remark 1.4. The relevance of Example 1.2 (4) to p-adic Hodge theory is as follows. Write C for the p-adic completion of $\mathbf{Q}_p(\mu_{p^\infty})$, the p-cyclotomic extension of $\mathbf{Q}_p$. Then the ring A described above coincides with Fontaine's $A_{\mathrm{inf}}(\mathcal{O}_C)$, and the element $\phi^{-1}(d)$ described above is a generator of Fontaine's θ-map $A_{\mathrm{inf}}(\mathcal{O}_C) \xrightarrow{\theta_{\mathcal{O}_C}} \mathcal{O}_C$. A similar remark applies to the generator of θ_R for any perfectoid ring R, as we shall see later.

Example 1.5 (The universal distinguished element)**.** There exists a universal δ-ring A equipped with a distinguished element d and such that $p \in \mathrm{Rad}(A)$. To see this, consider the free δ-ring $\mathbf{Z}\{d\}$ from Lemma II.2.5. By the structure of localizations, the δ-ring $A' := \mathbf{Z}_{(p)}\{d, \delta(d)^{-1}\}$ can be described explicitly as the localization $S^{-1}\mathbf{Z}_{(p)}\{d\}$, where $S = \{\delta(d), \phi(\delta(d)), \phi^2(\delta(d)), ...\}$. The desired δ-ring A is obtained by localizing A' along $\mathrm{Spec}(A'/p) \subset \mathrm{Spec}(A)$. In particular, the universal distinguished element d is a nonzerodivisor modulo p.

Our next goal is to show that the property that an element d of a δ-ring A is distinguished can be captured entirely in terms of the ideal (d) under mild hypotheses on the pair $(A, (d))$.

Lemma 1.6. *Let A be a δ-ring. Fix a distinguished element $f \in A$ and a unit $u \in A^*$. If $f, p \in \mathrm{Rad}(A)$, then uf is distinguished.*

Proof. We want to show that $\delta(uf)$ is a unit. Expanding, we have

$$\delta(uf) = u^p\delta(f) + f^p\delta(u) + p\delta(u)\delta(f).$$

The first term on the right side is a unit and the other two terms lie in $\mathrm{Rad}(A)$, so the whole expression is also a unit. □

The next lemma says that a distinguished element is "irreducible" in a certain sense.

Lemma 1.7 (Irreducibility of distinguished elements)**.** *Let A be a δ-ring with elements $f, g \in A$. Assume that g is distinguished and that we can write $g = fh$ for some $h \in A$. If $f, p \in \mathrm{Rad}(A)$, then f is distinguished and h is a unit.*

Proof. Applying δ to $g = fh$ gives

$$\delta(g) = f^p\delta(h) + h^p\delta(f) + p\delta(f)\delta(h).$$

The left side is a unit, while the first and last terms of the right side lie in $\mathrm{Rad}(A)$, so $h^p\delta(f)$ is a unit, which proves both claims. □

One can now characterize distinguished elements via their ideals.

Lemma 1.8. *Fix a δ-ring A and an element $f \in A$ such that $p, f \in \mathrm{Rad}(A)$. Then f is distinguished if and only if $p \in (f, \phi(f))$. In particular, the property that "f is distinguished" only depends on the ideal (f).*

Geometrically, the condition $p \in (f, \phi(f))$ says that the subschemes $Z := \mathrm{Spec}(A/f) \subset \mathrm{Spec}(A)$ and $\phi^{-1}(Z) := \mathrm{Spec}(A/\phi(f)) \subset \mathrm{Spec}(A)$ meet only in characteristic p.

Proof. Assume first that that f is distinguished, so $\delta(f)$ is a unit. Then the formula $\phi(f) = f^p + p\delta(x)$ immediately shows that $p \in (f, \phi(f))$.

Conversely, assume we can write $p = af + b\phi(f)$ for some $a, b \in A$. We want to show $\delta(f)$ is invertible. As $p, f \in \mathrm{Rad}(A)$, it suffices to show that $\delta(f)$ is invertible modulo (p, f) or equivalently that $A/(p, f, \delta(f)) = 0$. Assume this ring is not zero. Replacing A with its localization along

$V(p, f, \delta(f)) \subset \mathrm{Spec}(A)$, we may assume that $p, f, \delta(f) \in \mathrm{Rad}(A)$. Simplifying the equation $p = af + b\phi(f)$ using the definition of ϕ then yields an equation of the form $p(1 - b\delta(f)) = cf$ for suitable $c \in A$. As p is distinguished and $\delta(f) \in \mathrm{Rad}(A)$, the left hand side is distinguished by Lemma 1.6. But then Lemma 1.7 implies that f is distinguished, so $\delta(f)$ is a unit, which contradicts our hypothesis that $\delta(f) \in \mathrm{Rad}(A)$. □

For later applications, the following consequence characterizing ideals locally generated by distinguished elements will be quite useful.

Corollary 1.9. *Fix a δ-ring A and a Zariski locally principal ideal $I \subset A$ such that $(p, I) \subset \mathrm{Rad}(A)$. Then the following are equivalent:*

(1) $p \in (I, \phi(I))$.

(2) I is pro-Zariski locally generated by a distinguished element. More precisely, there exists a faithfully flat map $A \to A'$ of δ-rings where A' is a finite product of localizations of A along ϕ-stable multiplicative subsets and $IA' = (f)$ for a distinguished element f with $(p, f) \subset \mathrm{Rad}(A')$.

If these conditions are satisfied, then we have $p \in (I^p, \phi(I))$ in (1).

Proof. For (1) ⇒ (2): as I is Zariski locally principal, we may choose elements $g_1, ..., g_r \in A$ such that $(g_1, ..., g_r) = A$ and such that IA_{g_i} is principal. Write $B = \prod_{i=1}^r A_{g_i}$, so the map $A \to B$ is faithfully flat and $IB = (f)$ is principal. To get δ-structures, write A' for the localization of B along the closed set $V(p, f) \subset \mathrm{Spec}(B)$, so $(p, f) \subset \mathrm{Rad}(A')$. Then the composite $A \to B \to A'$ is still faithfully flat: the image of $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is stable under generalization by flatness, and contains $V(p, I) \subset \mathrm{Spec}(A)$ by construction, so it must be all of $\mathrm{Spec}(A)$ as $(p, I) \subset \mathrm{Rad}(A)$. Moreover, by construction, the ring A' is a finite product of localizations of A along ϕ-stable multiplicative subsets. As each such localization of A carries a unique compatible δ-structure, it follows that $A \to A'$ is naturally a faithfully flat map of δ-rings where A' is a finite product of localizations of A along ϕ-stable multiplicative subsets and such that $IA' = (f)$ for an element f with $(p, f) \subset \mathrm{Rad}(A')$. The distinguishedness of f follows from Lemma 1.8.

For (2) ⇒ (1): the condition $p \in (I, \phi(I))$ can be formulated as saying $p = 0 \in A/(I, \phi(I))$. The latter can be checked after faithfully flat base change, so we conclude using Lemma 1.8.

Assume now that (1) (and thus (2)) are satisfied. Then the construction above shows that A' can be chosen as an ind-Zariski localization of A. Moreover, the condition $p \in (I^p, \phi(I))$ can be checked after faithfully flat base change, we can assume $I = (f)$ for a distinguished element f. But then the equation $\phi(f) = f + p\delta(f)$ shows that $p \in (f^p, \phi(f))$ as $\delta(f)$ is a unit. □

In perfect p-adically complete δ-rings, distinguished elements admit a clean characterization via their Teichmuller expansions: the coefficient of p (i.e., the quantity "$\frac{\partial}{\partial p}(f)$" must be a unit).

Lemma 1.10 (Distinguished elements in perfect δ-rings)**.** *Let R be a perfect $\mathbf{F}_p$-algebra.*

(1) An element $d \in W(R)$ is distinguished if and only if the coefficient $a_1 \in R$ of p in the Teichmuller expansion $d = \sum_{i \geq 0} [a_i] p^i$ is a unit. (In the terminology of Fontaine [2], this says that d is primitive of degree 1.)

(2) Any distinguished element $d \in W(R)$ is a nonzerodivisor.

(3) For a distinguished element $d \in W(R)$, the p-power torsion in $W(R)/(d)$ is killed by p.

Proof. We only prove (1) and leave (2) and (3) as an exercise in using the δ-structure. Fix an element $d = \sum_{i \geq 0} [a_i] p^i \in W(R)$. Using the formula, $\delta(d) = \frac{\phi(d) - d^p}{p}$ and the fact that $\phi([a_0]) = [a_0^p]$, we immediately get $\delta(d) = [a_1^p] \mod p$. As A is p-adically complete, it follows that $\delta(d)$ is a unit exactly when $a_1 \in R$ is a unit, as wanted. □

2. Digresssion: recollections on derived completions

To work effectively with prisms, it will be useful to have a good theory of completions along an ideal. Unfortunately, the rings we shall encounter (such as perfect or perfectoid rings) are often non-noetherian, and the modules we shall encounter are often not finitely generated. In this setting, the classical theory of completions does not behave so well; for example, the cokernel of a map of complete modules can fail to be separated (and is thus not complete). This defect is rectified by passage to the derived variant of the notion of completeness.

Definition 2.1. Let A be a commutative ring, and let I be a finitely generated ideal. An A-complex $M \in D(A)$ is *derived I-complete* if for each $f \in I$, the derived inverse limit

$$T(M, f) := R\lim(\cdots \xrightarrow{f} M \xrightarrow{f} M \xrightarrow{f} M) \in D(A)$$

vanishes. This is equivalent to requiring that the natural map

$$M \to R\lim(M \otimes^L_{\mathbf{Z}[x]} \mathbf{Z}[x]/(x^n))$$

is a quasi-isomorphism, where on the right we treat M as a $\mathbf{Z}[x]$-module via restriction of scalars along $\mathbf{Z}[x] \xrightarrow{x \mapsto f} A$. An A-module M is called *derived I-complete* if it is so when regarded as a complex. We say that an A-module M is *classically I-complete* if $M \simeq \lim_n M/I^n M$.

It is an elementary fact that the derived I-completeness of M can be checked by simply checking that $T(M, f) = 0$ for f in a generating set for I. For finitely generated modules over noetherian rings, the above notion is equivalent to the classical one (see (5) below). We list a few standard properties; all assertions below (and more) can be found in [3, Tag 091N].

(1) The collection of all derived I-complete A-complexes forms a full triangulated subcategory of $D(A)$ closed under derived inverse limits. This inclusion has a left-adjoint $M \mapsto \widehat{M}$ whose unit map can be described as follows: if $I = (f_1, ..., f_r)$, then for any $M \in D(A)$, the natural map

$$M \to \widehat{M} := R\lim_n \left(M \otimes^L_{\mathbf{Z}[x_1,...,x_r]} \mathbf{Z}[x_1, ..., x_r]/(x_1^n, ..., x_r^n)\right)$$

(where x_i acts by f_i on M) is the universal map from M into a derived I-complete A-complex; we call $\widehat{M}$ the derived I-completion of the A-complex M. This formula also shows that if $M \in D^{[a,b]}(A)$, then $\widehat{M} \in D^{[a-r,b]}(A)$. In particular, the derived completion functor preserves $D^{\leq 0}(A)$.

Example 2.2. Say $A = \mathbf{Z}$ and $I = (p)$. If $M = \mathbf{Q}/\mathbf{Z}$, then $\widehat{M} \simeq \mathbf{Z}_p[1]$.

(2) (Derived Nakayama): A derived I-complete A-complex $M \in D(A)$ (resp. A-module N) is 0 if and only if $M \otimes^L_A A/I \simeq 0$ (resp. $N/IN = 0$).

(3) An object $M \in D(A)$ is derived I-complete if and only if each $H^i(M)$ is so. It then follows from (1) that for an A-module M, the natural map $M \to H^0(\widehat{M})$ is the universal map from M to a derived I-complete A-module. We thus call $H^0(\widehat{M})$ the derived I-completion of the A-module M (as a module); note that the derived I-completion of M when regarded as an A-complex may differ from the derived I-completion of M when regarded as an A-module.

(4) The collection of all derived I-complete A-modules forms an abelian subcategory of all A-modules closed under the formation of kernels, cokernels and images. In particular, finitely presented modules over a derived I-complete ring are themselves derived I-complete. (Note that this property is false for classically complete modules over a classically complete ring.) For future reference, we remark that it follows from (3) that the inclusion of derived I-complete A-modules inside all A-modules has a left-adjoint given by $M \mapsto H^0(\widehat{M})$.

Exercise 2.3. Let $M \to N$ be a map of derived I-complete A-modules. If $M/IM \to N/IM$ is surjective, show that $M \to N$ is surjective.

(5) If A-module M is classically I-complete, then M is also derived I-complete; the converse holds true provided M is I-adically separated.

The derived complete modules that we shall encounter will often by classically complete. The reason for this usually will be the following lemma.

Lemma 2.4. *Let A be a commutative ring, and let $f \in A$ be an element. If an A-module M has bounded f^∞-torsion (i.e., there exists some $n > 0$ such that $M[f^\infty] = M[f^n]$), then derived f-completion of M as a complex is discrete and coincides with its classical f-completion.*

In particular, for an f-torsionfree A-module, the derived f-completeness of M is equivalent to classical f-completeness.

Proof. The derived f-adic completion $\widehat{M}$ of M (as a complex) is defined as $R\lim_n(M \otimes^L_{\mathbf{Z}[x]} \mathbf{Z}[x]/(x^n))$, where x acts by f. There is a standard resolution of

$$\left(\mathbf{Z}[x] \xrightarrow{x^n} \mathbf{Z}[x]\right) \to \mathbf{Z}[x]/(x^n)$$

that helps compute the derived tensor product appearing in the definition of $\widehat{M}$. As the derived inverse limit functor has cohomological dimension 1, we get the following:

- $H^i(\widehat{M}) = 0$ for $i \neq -1, 0$.
- $H^{-1}(\widehat{M}) = \lim_n M[f^n]$,where the transition maps are multiplication by f.
- There is a short exact sequence

$$0 \to R^1 \lim M[f^n] \to H^0(\widehat{M}) \to \lim_n M/f^n M \to 0.$$

Now our hypothesis $M[f^\infty] = M[f^n]$ for $n \gg 0$ implies that the projective system $\{M[f^n]\}$ is essentially 0, i.e., any sufficiently large composition of transition maps is 0. It immediately follows that $\lim_n M[f^n] = R^1 \lim M[f^n] = 0$, so the above discussion then shows that $\widehat{M} \simeq H^0(\widehat{M}) \simeq \lim_n M/f^n M$, as wanted. □

In the context of large non-noetherian rings, a completion of a flat module can easily fail to be flat. To circumvent this problem, we shall work with the following softer notion.

Definition 2.5. Let A be a commutative ring, and let $I \subset A$ be a finitely generated ideal. We say that an A-complex M is *I-completely flat* if $M \otimes^L_A A/I$ is a flat A/I-module (i.e., has cohomology only in degree 0 where it is given by a flat A/I-module).

It is elementary to see that I-complete flatness is equivalent to I^n-complete flatness for any $n \geq 1$. Moreover, any flat A-module M is clearly I-completely flat. The advantage of the above notion, however, is the following observation:

Lemma 2.6. *Let A be a commutative ring, and let $I \subset A$ be a finitely generated ideal. Then the derived I-completion of a flat A-complex M is I-completely flat.*

Proof. Using the universal property of the derived completion map $M \to \widehat{M}$, and the observation that any A/I-complex is derived I-complete, one immediately shows $M \otimes^L_A A/I \simeq \widehat{M} \otimes^L_A A/I$. □

We warn the reader that the derived I-completion of a flat A-module M, when regarded as an A-complex, might fail to be discrete. In fact, this may happen for $M = A$ itself. Lemma 2.4 gives one example of a situation where this pathology does not occur.

Exercise 2.7. Let A be a δ-ring, and let $I \subset A$ be an ideal containing p. Then the derived I-completion $\widehat{A}$ of A (as a module) admits a unique δ-structure compatible with the one on A. (Hint: if $A \to W_2(A)$ denotes the map classifying the δ-structure on A, then show that the composite $A \to W_2(A) \to W_2(\widehat{A})$ factors uniquely over $A \to \widehat{A}$ using the universal property.)

3. Prisms

The category of δ-pairs is formed by pairs (A, I) where A is a δ-ring and $I \subset A$ is an ideal; morphisms $(A, I) \to (B, J)$ in this category are simply δ-maps $A \to B$ that carry I into J.

Definition 3.1 (The category of prisms)**.** We shall study the following types of δ-pairs.

1. A δ-pair (A, I) is a *prism* if $I \subset A$ defines an Cartier divisor on $\mathrm{Spec}(A)$ such that A is derived (p, I)-complete, and $p \in I + \phi(I)A$. The category of prisms is the corresponding full subcategory of all δ-pairs.
2. A map $(A, I) \to (B, J)$ of prisms is *(faithfully) flat* if the map $A \to B$ is (p, I)-completely (faithfully) flat, i.e., $A/(p, I) \to B \otimes^L_A A/(p, I)$ is (faithfully) flat.
3. A prism (A, I) is called
   - *perfect* if A is a perfect δ-ring, i.e., $\phi : A \to A$ is an isomorphism.
   - *bounded* if A/I has bounded p^∞-torsion, i.e., $A/I[p^\infty] = A/I[p^c]$ for some $c \geq 0$.
   - *crystalline* if $I = (p)$.

Remark 3.2. The prismatic condition appearing in (1) above has a clean geometric interpretation: the condition $p \in (I, \phi(I))$ says that the closed subschemes $\phi^{-1}V(I)$ and $V(I)$ of $\mathrm{Spec}(A)$ meet only in characteristic p, i.e., the intersection does not "stick out" towards characteristic 0.

Example 3.3. Let us record some examples.

1. For any p-torsionfree and p-adically complete δ-ring A, the pair $(A, (p))$ is a bounded prism.
2. Each of the pairs (A, d) Example 1.2 gives a bounded prism $(A, (d))$.
3. Let $A_0 = \mathbf{Z}_{(p)}\{\xi, \delta(\xi)^{-1}\}$ be the displayed localization of the free δ-ring on a variable ξ. Let A be the (p, ξ)-completion of A_0, and let $I = (\xi) \subset A$. Then the pair (A, I) is a bounded prism, and the element $\xi \in I$ is a distinguished generator.
4. We will see later that a perfectoid ring gives a perfect prism in an essentially unique way.

The boundedness of a prism is a somewhat technical condition, but will be satisfied in essentially all examples we encounter. The reason this condition is useful is the following.

Exercise 3.4. Let (A, I) be a bounded prism. Let M be a flat A-module. Then the derived (p, I)-completion of M as a complex is discrete, classically (p, I)-complete, and (p, I)-completely flat as an A-complex.

Lemma 3.5. *Let (A, I) be a prism. Then the ideal $\phi(I)A \subset A$ is principal and any generator is a distinguished element. In particular, if (A, I) is a perfect prism, then $I = (f)$ for a distinguished element f.*

Proof. Once we know one generator of $\phi(I)A$ is distinguished, it follows from Lemma 1.7 that all generators are distinguished. So it suffices to show that $\phi(I)A$ is generated by a distinguished element. By hypothesis and Corollary 1.9, we can write $p = a + b$ where $a \in I^p$ and $b \in \phi(I)A$. We claim that b generates $\phi(I)A$, i.e., the map $A \to \phi(I)A$ defined by $1 \mapsto b$ is surjective. Choose a faithfully flat map $A \to A'$ as in Corollary 1.9. It is enough to show that the map $A' \to \phi(I)A'$ induced by $1 \to b$ is surjective. We have $IA' = (f)$ for a distinguished element $f \in A'$. Since $a \in I^p$ and $b \in \phi(I)$, we can write $a = xf^p$ and $b = y\phi(f)$ for suitable $x, y \in A'$. Our task is to show that

y is a unit. As $p, f \in \mathrm{Rad}(A')$, it suffices to show that y is a unit modulo (p, f) or equivalently that $A'/(p, f, y) = 0$. If not, by localizing along $\mathrm{Spec}(A'/(p, f, y)) \subset \mathrm{Spec}(A')$, we may assume that $p, f, y \in \mathrm{Rad}(A')$. The equation $p = a + b = xf^p + y\phi(f)$ simplifies to show

$$p(1 - y\delta(f)) = f^p(x + y) = f \cdot (f^{p-1}(x + y))$$

Now $1 - y\delta(f)$ is a unit as $y \in \mathrm{Rad}(A')$, so the left side is distinguished by Lemma 1.6. Lemma 1.7 then implies that $f^{p-1}(x + y)$ is a unit, whence f is a unit (as $p - 1 \geq 1$), which contradicts $f \in \mathrm{Rad}(A')$. □

Exercise 3.6. Let (A, I) be a prism. Show that I defines a p-torsion element of $\mathrm{Pic}(A)$.

The following lemma significantly simplifies the study of maps between prisms.

Lemma 3.7 (Rigidity of maps)**.** *Let $(A, I) \to (B, J)$ be a map of prisms. Then $I \otimes_A B \simeq J$ via the natural map. In particular, $IB = J$.*

Conversely, if B is a derived (p, I)-complete δ-A-algebra, then (B, IB) is a prism if and only if $B[I] = 0$.

Thus, the forgetful functor $(B, IB) \mapsto B$ from prisms over (A, I) to δ-A-algebras is fully faithful.

Proof. It suffices to show that $I \otimes_A B \to J$ is surjective (or equivalently that $IB = J$): any surjection of invertible modules over a commutative ring is an isomorphism. By Corollary 1.9, we can choose a faithfully flat map $A \to A'$ of δ-rings such that $IA' = (f)$ for a distinguished element $f \in A'$ and such that $(p, IA') \subset \mathrm{Rad}(A')$. Write B' for the localization of the base change $A' \otimes_A B$ along $V(p, J)$. Then $B \to B'$ is a faithfully flat map of δ-rings as well. By passing to a further faithfully flat ind-Zariski localization of B', we can also ensure that $JB' = (g)$ is generated by a distinguished element $g \in B'$. Now $IB' \subset JB'$ is an inclusion of ideals generated by distinguished elements, and is thus an equality by Lemma 1.7. It follows by faithful flatness of $B \to B'$ that $IB \subset J$ must be an equality as well.

For the second statement, fix a derived (p, I)-complete δ-A-algebra. Note that $B[I] = 0$ if and only if $I \otimes_A B \to IB$ is an isomorphism. Now if (B, IB) is a prism, then $I \otimes_A B \to IB$ must be an isomorphism (by virtue of being a surjection between invertible B-modules). Conversely, if this map is an isomorphism, then $IB \subset B$ is an invertible B-module, and it is then immediate that (B, IB) is a prism. □

References

[1] B. Bhatt, M. Morrow, P. Scholze, *Integral p-adic Hodge theory*

[2] J. M. Fontaine, *Perfectoides, presque purete et monodromie-poids*, Bourbaki talk.

[3] The Stacks Project