LECTURE XI: q-CRYSTALLINE COHOMOLOGY

In this lecture, we construct a q-crystalline site $(R/\mathbf{Z}_p[\![q-1]\!])_{\mathrm{qcrys}}$ for smooth $\mathbf{Z}_p$-algebras R by defining a suitable notion of q-PD thickenings. The resulting q-crystalline cohomology theory $q\Omega_R$ is then shown to have two properties: it is identified with the prismatic cohomology of a certain twist $R^{(1)}$ of R, and it is computed (after choosing étale co-ordinates) by the q-de Rham complexes discussed in Lecture X. Combining the two, we obtain a concrete complex computing the prismatic cohomology of $R^{(1)}$; up to a Frobenius twist, this is analogous to computing the crystalline cohomology of a smooth $\mathbf{Z}_p$-algebra R as the de Rham cohomology of a lift of R to $\mathbf{Z}_p$.

The following notation will be used throughout this lecture.

Notation 0.1. We view $A := \mathbf{Z}_p[\![q-1]\!]$ as as δ-ring via $\delta(q) = 0$. Unless otherwise specified, the ring $\mathbf{Z}_p$ is viewed as a δ-A-algebra via the δ-map $A \xrightarrow{q \mapsto 1} \mathbf{Z}_p$. We shall use without comment the fact that the ideals $(p, q-1)$ and $(p, [p]_q)$ define the same topology on A, and hence derived $(p, [p]_q)$-completion coincides with derived $(p, q-1)$-completion. We shall also write $\mathbf{Z}_p[\epsilon_p] = A/([p]_q)$, so $q = \epsilon_p$ is a primitive p-th root of unity.

1. q-PD THICKENINGS

To construct the q-crystalline site, we need a q-deformation of the notion of divided power thickenings. The map $A \to A/(q-1) \simeq \mathbf{Z}_p$ should be the basic example of such a thickening, so the naive definition of divided powers certainly does not work. While we still do not know how to define this notion in the generality of all (p-complete) commutative rings, the following definition yields a good answer if one restricts attention to δ-rings (which suffices for our purposes).

Definition 1.1 (q-PD-thickenings)**.** A *q-PD pair* is a pair (D, I) where D is a δ-A-algebra and $I \subset D$ is an ideal such that the following hold:

(1) Both D and D/I are $(p, [p]_q)$-complete.

(2) D is $[p]_q$-torsionfree.

(3) I contains $q-1$ and we have $\phi(I) \subset [p]_q D$.

There is an evident category of q-PD pairs. Given a q-PD pair (D, I), we often call the corresponding map $D \to D/I$ a *q-PD thickening.*

Remark 1.2. For relative variants of the q-crystalline theory, it is convenient to impose an additional technical condition in Definition 1.1: one requires $D/(q-1)$ to have finite Tor-amplitude over D. As we do not consider the relative theory in these lectures, we can avoid this complication.

Example 1.3. Some important examples include:

(1) (The flat case) The pair $(A, (q-1))$ is a q-PD-pair. In fact, this is the initial object in the category of all q-PD pairs[1].

More generally, if D is any $(p, [p]_q)$-completely flat δ-A-algebra, then $(D, (q-1))$ is a q-PD-pair. An important example of such a D for our purposes is provided by $S[\![q-1]\!]$, where $(S, \square)$ is a framed pair (Definition X.1.5), and $S[\![q-1]\!]$ is given the unique δ-structure compatible with the $(p, [p]_q)$-completely flat map $\widetilde{\square}$ (Exercise X.1.6) and the requirement that $\delta(x_i) = 0$ for all i.

[1] Note that $q-1$ is not distinguished element of the δ-ring A, so the pair $(A, (q-1))$ is not a prism.

(2) (The $q=1$ case) Let D be a p-torsionfree and p-adically complete δ-ring, regarded as a A-algebra via $q \mapsto 1$. Given a p-complete ideal $I \subset D$, one checks using Lemma VI.2.1 that (D,I) is a q-PD-pair exactly when the divided powers of I lie in D, i.e., $\frac{x^n}{n!} \in D$ for all $x \in I$ and $n \geq 0$. (Note that it need not be the case that $\frac{x^n}{n!} \in I$ for all n.)

(3) (The perfect case) Let B be a perfect $(p,[p]_q)$-complete δ-A-algebra. Then $[p]_q \in B$ is a distinguished element and thus a nonzerodivisor (Lemma III.1.10). Let $I \subset B$ be any $(p,[p]_q)$-complete ideal. It is then easy to see that (B,I) gives a q-PD-pair exactly when $I \subset \phi^{-1}([p]_q)B$, where we recall $\phi^{-1}([p]_q) = \frac{q-1}{q^{1/p}-1}$. In particular, the pair $(B,(\phi^{-1}([p]_q)))$ is both a q-PD pair and a prism.

A more geometric example is recorded in Example 1.7.

Next, we need to explain the construction of q-PD envelopes (i.e., the q-analog Construction VI.1.1). To ensure that this operation is not too wild, we need the following crucial lemma (see Remark 1.5 to understand what classical fact is being q-deformed).

Lemma 1.4. *Let D be a $[p]_q$-torsionfree $(p,[p]_q)$-complete δ-A-algebra. Let $f \in D$ be an element such that $\phi(f) \in [p]_qD$. Then $\phi(\frac{\phi(f)}{[p]_q} - \delta(f)) \in [p]_qD$.*

Proof sketch. It is enough to prove the statement in the universal case. One then shows using Corollary VI.2.3 that the universal such D is flat over A (argument omitted). In particular, D is $\phi([p]_q)$-torsionfree. Our goal is to show that

$$\frac{\phi^2(f)}{\phi([p]_q)} \equiv \phi(\delta(f)) \mod [p]_qD.$$

Note that $\phi([p]_q) = \frac{q^{p^2}-1}{q^p-1}$, which is congruent to p modulo $[p]_q$. As $A/([p]_q)$ is p-torsionfree, the same holds true for $D/([p]_q)$ by flatness. We are therefore reduced to checking that

$$\phi^2(f) \equiv \phi([p]_q)\phi(\delta(f)) \mod [p]_qD.$$

Noting again that $\phi([p]_q) = p \mod [p]_qD$, this is equivalent to showing that

$$\phi^2(f) \equiv p\phi(\delta(f)) \mod [p]_qD.$$

But thus follows by applying ϕ to $\phi(f) = f^p + p\delta(f)$ and using that $\phi(f^p) \in [p]_qD$ since we already have $\phi(f) \in [p]_qD$ by hypothesis. □

Remark 1.5 (The $q=1$ limit)**.** Assume we are in the setting of Lemma 1.4 and that $q=1$ on D. In this case, the expression $\frac{\phi(f)}{[p]_q} - \delta(f)$ coincides with $\frac{f^p}{p}$. Using the fact that $\phi(x) \equiv x^p \mod pD$ for any x, one can translate Lemma 1.4 to the following assertion:

($*$) Fix a p-torsionfree and p-complete δ-ring D. If $f \in D$ with $f^p \in pD$, then $f^{p^2} \in p^{p+1}D$.

This is exactly the assertion ($*$) in the proof of Lemma VI.2.1 (but with a different proof).

Using the previous lemma, we can construct q-PD-envelopes in certain situations. These are explained in the next proposition, and may be summarized as follows: the notion of divided power envelopes of a smooth affine scheme inside (say) affine space has a good q-deformation.

Proposition 1.6 (Existence of q-PD-envelopes)**.** *Let R be a formally smooth $\mathbf{Z}_p$-algebra. Let P be a δ-A-algebra equipped with a surjection $P \to R$ with kernel J. Assume that P is formally smooth as a A-algebra. Then there is a universal map $(P,J) \to (D,I)$ of δ-pairs with (D,I) being a q-PD pair. Moreover, this construction has the following properties:*

(1) D is $(p,[p]_q)$-completely flat over A.

(2) The natural map gives an isomorphism $P/J \simeq D/I$, so the resulting map $D \to D/I \simeq R$ is a q-PD-thickening.

(3) The ring $D/(q-1)$ identifies with the p-completed divided power envelope of the surjection $P/(q-1) \to R$ (see Construction VI.1.1).

We often write $D_{J,q}(P) := D$, and abusively call it the q-PD envelope of (P, J).

Proof sketch. Assume for simplicity that the kernel of $P \to R$ is finitely generated. In this case, it is generated by a sequence of the form $(q-1, x_1, ..., x_r)$, where $x_1, ..., x_r$ give a sequence that is regular on $P/(p, q-1)$. In this case, set $D := P\{\frac{\phi(x_1)}{[p]_q}, ..., \frac{\phi(x_r)}{[p]_q}\}^\wedge$, i.e., the $(p, [p]_q)$-complete δ-P-algebra obtained by freely adjoining $\frac{\phi(x_i)}{[p]_q}$ for $i = 1, ..., r$.

One then proves (1) using Corollary VI.2.3. In particular, we obtain the presentation $D/(q-1) \simeq P/(q-1)\{\frac{\phi(x_1)}{p}, ..., \frac{\phi(x_r)}{p}\}^\wedge$, where the completion is derived p-adic.

Next, for (3), note that the formation of D commutes with base change along $P \to P/(q-1)$ by the last sentence of the first paragraph of this proof. We may then deduce (3) using Lemma VI.2.1. In particular, this gives a map $D \to D/(q-1) \to R$. Let I denote the kernel, so I is simply the preimage of kernel of the PD-thickening $D/(q-1) \to R$ under $D \to D/(q-1)$.

We next check that (D, I) is a q-PD pair. Thus, we must check that for any $y \in I$, we have $\phi(y) \in [p]_q D$. Let $I' = \phi^{-1}([p]_q D) \cap I$, so we must show $I' = I$. We already know that $q-1, x_1, ..., x_r \in I'$ by construction. Moreover, Lemma 1.4 implies that if $y \in I'$, then $\nu(y) := \frac{\phi(y)}{[p]_q} - \delta(y) \in \phi^{-1}([p]_q D)$. It is also easy to checking using (3) that $\nu(y) \in I$ for $y \in I'$, and thus $I' \subset I$ is stable under the operation $\nu(-)$. Also, I' is $(p, [p]_q)$-complete by construction. To check that $I' = I$, it suffices to show they agree modulo $q-1$. The claim then follows from (3) and the fact that the kernel of PD-thickening $D/(q-1) \to R$ (which is simply the image of I) is the smallest p-complete ideal that contains each x_i and is stable under the operation $y \mapsto \frac{y^p}{p}$ (see proof of [5, Tag 07GS]).

The universality of $(P, J) \to (D, I)$ is left to the reader. ☐

The following example indicates the kind of denominators one is allowed when working with q-PD envelopes that arise most naturally in practice.

Example 1.7 (The q-PD envelope of the diagonal in $\mathbf{A}^1 \times \mathbf{A}^1$)**.** Consider $P = \mathbf{Z}_p[q, x, y]^\wedge_{(p,q-1)}$, viewed as a δ-ring via $\delta(q) = \delta(x) = \delta(y) = 0$. Let $J = (x-y) \subset P$ be the ideal of the "diagonal". Then the q-PD envelope $D_{J,q}(P) := P\{\frac{x^p - y^p}{[p]_q}\}^\wedge$ turns out to be topologically free over $\mathbf{Z}_p[q, x]$ (or $\mathbf{Z}_p[q, y]$) with a basis given by the q-divided powers

$$\gamma_{k,q}(x-y) := \frac{(x-y)(x-qy)....(x-q^{k-1}y)}{[k]_q!}$$

for $0 \leq k < \infty$, where $[k]_q! = \prod_{i=1}^k [i]_q$.[2] This claim follows essentially from calculations of Pridham [3, Lemma 1.5]. Let us only explain here why $\gamma_{p,q}(x-y)$ lies in $D_{J,q}(P)$. To see this, since $[k]_q$ is invertible for $p \nmid k$, it is enough show that

$$[p]_q \mid (x-y)(x-qy)...(x-q^{p-1}y)$$

in $D_{J,q}(P)$. For this, we claim that there is a congruence

$$(x-y)(x-qy)...(x-q^{p-1}y) = x^p - y^p \mod [p]_q P$$

[2] Note that $\gamma_{k,q}(x-y)$ specializes to $\gamma_k(x-y)$ when $q = 1$, so this description is consistent Proposition 1.6 (3) and the usual description of divided power envelopes.

in P. Granting this, the claim follows since $[p]_q$ divides $x^p - y^p = \phi(x-y)$ in $D_{J,q}(P)$ by construction. The congruence above amounts to the identity

$$\prod_{i=0}^{p-1}(x - \epsilon_p^i y) = x^p - y^p$$

in the ring $\mathbf{Z}_p[\epsilon_p, x, y]^\wedge$, which is easy to see.

2. The q-crystalline site

Having defined a notion of a q-PD thickening, it is relatively straightforward to define a q-crystalline site.

Definition 2.1 (The q-crystalline site)**.** Let R be a formally smooth $\mathbf{Z}_p$-algebra. We define the *q-crystalline site* $(R/A)_{\mathrm{qcrys}}$ to be the category of triples (D, I, η), where (D, I) is a q-PD pair and $\eta : D/I \simeq R$ is an isomorphism. We topologize this category via the indiscrete topology, so every presheaf is a sheaf.

The assignment $(D, I) \mapsto D$ defines a sheaf $\mathcal{O}_{\mathrm{qcrys}}$ on $(R/A)_{\mathrm{qcrys}}$, and we set

$$q\Omega_R := R\Gamma((R/A)_{\mathrm{qcrys}}, \mathcal{O}_{\mathrm{qcrys}}).$$

By construction, this is a commutative algebra in $\mathcal{D}_{comp}(A)$ that comes equipped with a Frobenius endomorphism ϕ_R (induced by the Frobenius map of the δ-rings in $(R/A)_{\mathrm{qcrys}}$).

Recall that in Construction VI.1.4, we constructed Cech-Alexander complexes computing crystalline cohomology using the notion of divided power envelopes. An analogous construction, using the q-PD envelopes provided by Proposition 1.6 instead of the usual ones, yields functorial Cech-Alexander type complexes computing q-crystalline cohomology:

Construction 2.2 (Cech-Alexander complexes for q-crystalline cohomology)**.** Let R be a formally smooth $\mathbf{Z}_p$-algebra. Choose a surjection $P \to R$ where P is the $(p, [p]_q)$-completion of a free δ-A-algebra. Let $P^\bullet$ be the $(p, [p]_q)$-completed Cech nerve of $A \to P$, so each P^n is the $(p, [p]_q)$-completion of a free δ-A-algebra. For each $n \geq 0$, we have a surjection $P^n \xrightarrow{\mu} P \to R$ with kernel J^n, where the first map is the multiplication map. As each P^n is formally smooth as a $\mathbf{Z}_p[\![q-1]\!]$-algebra, we may apply Proposition 1.6 to the cosimplicial δ-pair $(P^\bullet, J^\bullet)$, we obtain a cosimplicial diagram

$$C^\bullet_{\mathrm{qcrys}}(R/A) := D_{J^\bullet,q}(P^\bullet) := \Big(D_{J^0,q}(P^0) \rightrightarrows D_{J^1,q}(P^1) \substack{\rightarrow\\\rightarrow\\\rightarrow} D_{J^2,q}(P^2) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \Big)$$

of q-PD pairs. Moreover, by Proposition 1.6, we may regard this as a cosimplicial object of $(R/A)_{\mathrm{qcrys}}$. As P was free, it follows by category theory (Lemma V.4.3) that $D_{J^\bullet,q}(P^\bullet)$ computes the q-crystalline cohomology complex $q\Omega_R$. As in all previous iterations of such constructions (such as V.5.3 or VI.1.4), we can make the construction $R \mapsto D_{J^\bullet,q}(P^\bullet)$ strictly functorial in R by choosing the initial surjection $P \to R$ functorially in R.

We can now begin proving comparison theorems that help understand q-crystalline cohomology. First, we discuss the $q = 1$ specialization.

Theorem 2.3 (Comparison with de Rham cohomology)**.** *Let R be a formally smooth $\mathbf{Z}_p$-algebra. Then there is a canonical identification*

$$q\Omega_R \otimes^L_A \mathbf{Z}_p \simeq \Omega^*_{R/\mathbf{Z}_p}$$

of commutative algebras in $\mathcal{D}_{comp}(\mathbf{Z}_p)$.

Proof sketch. Instead of constructing an isomorphism with de Rham cohomology on the right, we construct an isomorphism with the crystalline cohomology of $\overline{R} := R/p$ relative to $\mathbf{Z}_p$; this suffices by the crystalline-de Rham comparison (Theorem VI.1.6). Thus, we must explain why $q\Omega_R \otimes_A^L \mathbf{Z}_p$ computes $R\Gamma_{crys}(\overline{R}/\mathbf{Z}_p)$. With notation as in Construction 2.2, the left side is computed by $C^\bullet_{\mathrm{qcrys}}(R/A)/(q-1)$: this follows as $-\otimes_A^L \mathbf{Z}_p$ has finite homological dimension and thus commutes with totalizing a cosimplicial A-module. By Proposition 1.6 (3), this coincides[3] with the cosimplicial ring $C^\bullet_{crys}(\overline{R}/\mathbf{Z}_p)$ arising from the Cech-Alexander complex construction for crystalline cohomology (Construction 2.2), so we are done. ☐

Remark 2.4 (Smaller Cech-Alexander complexes)**.** It follows from Theorem 2.3 one can use any formally smooth A-algebra P (not merely the formally free ones) whilst defining the Cech-Alexander complex as in Construction 2.2 without changing the cohomology. Indeed, the Cech-Alexander complex defined using a formally free algbera maps to the Cech-Alexander complex defined using any formally smooth algebra, and this map is a quasi-isomorphism: it is so modulo $(q-1)$ by Theorem 2.3 and a well-known property of crystalline cohomology. A particularly convenient such choice is obtained by taking P to be a formally smooth lift of R across $A \to \mathbf{Z}_p$.

Next, we explain how q-crystalline cohomology relates to prismatic cohomology.

Theorem 2.5 (Comparison with prismatic cohomology)**.** *Let R be a formally smooth $\mathbf{Z}_p$-algebra and let $R^{(1)} := R \otimes_{\mathbf{Z}_p} \mathbf{Z}_{[}\epsilon_p]$. Computing prismatic cohomology $R^{(1)}$ relative to the prism $(A, ([p]_q))$, we have a canonical identification*

$$\Delta_{R^{(1)}/A} \simeq q\Omega_R$$

of commutative algebras in $\mathcal{D}_{comp}(A)$.

Proof sketch. Let us first explain how to construct a map $\Delta_{R^{(1)}/A} \to q\Omega_R$; this is analogous to the discussion following Theorem VI.3.2. It is enough to explain why each q-PD pair $(D, I) \in (R/A)_{\mathrm{qcrys}}$ yields an object $(R^{(1)} \to D/([p]_q) \leftarrow D) \in (R^{(1)}/A)_\Delta$. Any such (D, I) satisfies $\phi(I) \subset [p]_q D$ by hypothesis, and thus we have an induced map $R \simeq D/I \to D/([p]_q)$ of rings. This map is linear over the Frobenius on D and hence also over the Frobenius on A. Linearizing, we obtain an A-algebra map $A \otimes_{\phi,A} R \to D/([p]_q)$, where the A-algebra structure on the left is defined by the first copy of A. As $[p]_q = 0$ on the target, this factors through an A-algebra map $(A \otimes_{\phi,A} R)/([p]_q) \to D/([p]_q)$. But the left side is exactly $R^{(1)}$, so we obtain the desired object of $(R^{(1)}/A)_\Delta$.

It remains to prove that the map constructed above is an isomorphism. As both sides are $(p, [p]_q)$-complete A-complexes, it is enough to prove the same after base change along $A \xrightarrow{q \mapsto 1} \mathbf{Z}_p$ by the derived Nakyama lemma. After this base change, thanks to Theorem 2.3 and the base change compatibility of prismatic cohomology, the claim reduces to the crystalline comparison theorem for prismatic cohomology (Theorem VI.3.2); details omitted. ☐

As mentioned in Remark X.1.14, we can now specialize at a primitive p-th root of unity to obtain a "Cartier isomorphism" (which was constructed directly in [4, Proposition 3.4]).

Corollary 2.6 (Hodge-Tate comparison for q-crystalline cohomology)**.** *Let R be a formally smooth $\mathbf{Z}_p$-algebra. Then there is a natural identification*

$$H^*(q\Omega_R \otimes_A^L A/([p]_q)) \simeq \Omega^*_{R/\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\epsilon_p]$$

of graded algebras over $A/([p]_q) \simeq \mathbf{Z}_p[\epsilon_p]$.

[3]We implicitly use that the p-adically completed divided power envelopes of $P/(q-1) \to R$ and $P/(q-1) \to R \to \overline{R}$ are the same as we always require our divided powers to be compatible with those on (p).

Proof. This follows from the comparison of q-crystalline and prismatic cohomology (Theorem 2.5), the Hodge-Tate comparison for prismatic cohomology (Theorem VI.0.1), and the fact that the formation of differential forms commutes with base change. $\square$

Finally, it remains to explain why q-crystalline cohomology can be computed by a q-de Rham complex. Our argument here is analogous to the one in [1] used to relate crystalline cohomology to the de Rham complex. To run this argument, it is important to have a notion of q-de Rham complexes not just for formally smooth $\mathbf{Z}_p$-algebras R equipped with an étale map from a p-complete polynomial ring, but actually also for those presented as quotients of p-complete polynomial rings. We discuss this extension next.

Construction 2.7 (q-de Rham complex of a q-PD envelope)**.** Let P be a $(p,[p]_q)$-complete A-algebra that is formally étale over $A[x_1,...,x_n]$, so P carries a unique δ-A-algebra structure determined by $\delta(x_i)=0$. We may regard P as the unique deformation to A of the framed $\mathbf{Z}_p$-algebra $P/(q-1)$ with the framing $\square$ determined by $x_1,...,x_n$ (see Definition X.1.5 and Exercise X.1.6). Let $\gamma_i : P \to P$ and $\nabla_{q,i}$ be as in Construction X.1.7.

Let R be a formally smooth $\mathbf{Z}_p$-algebra, and assume we are given a surjection $P \to R$ with kernel J. Let $D = D_{J,q}(R)$ be the q-PD envelope provided by Proposition 1.6. We claim that for each $1 \le i \le n$, the q-derivatives $\nabla_{q,i}$ on P (Construction X.1.7) extend to commuting endomorphisms $\nabla_{q,i}$ of D, thus yielding a q-de Rham complex[4] $q\Omega^*_{D,\square} := \mathrm{Kos}(D; \nabla_{q,1}, ..., \nabla_{q,n})$ for D. To see this, the key step is the following:

Lemma 2.8. *For each $1 \le i \le n$, the automorphism γ_i of P extends uniquely to an automorphism γ_i of D that is congruent to the identity modulo $qx_i - x_i$.*

Proof sketch. It is enough to show that each γ_i extends to uniquely to an endomorphism of D that is congruent to the identity modulo $qx_i - x_i$: any endomorphism of D that is congruent to the identity modulo a topologically nilpotent element (such as $qx_i - x_i$) must be an isomorphism.

Consider the composition $P \xrightarrow{\gamma_i} P \xrightarrow{can} D$. It is enough to show that this composition factors uniquely over $P \xrightarrow{can} D$. As all maps in sight are maps of δ-rings, we may use the universal property of $P \xrightarrow{can} D$ (Proposition 1.6) and the $[p]_q$-torsionfreeness of D to reduce to checking the following:

($*$) For $f \in J$, we have $\phi(\gamma_i(f)) \in [p]_q D$ and $\frac{\phi(\gamma_i(f))}{[p]_q} \equiv \frac{\phi(f)}{[p]_q} \mod (qx_i - x_i)D$.

As the automorphism γ_i of P is congruent to the identity modulo $qx_i - x_i$, we can write

$$\gamma_i(f) = f + (q-1)x_i g$$

for some $g \in P$. Applying ϕ and dividing (in D) by $[p]_q$ gives ($*$). $\square$

One also checks that the automorphisms $\{\gamma_i\}_{i=1,...,n}$ of D constructed in the above lemma give a family of commuting endomorphisms of D. One can now imitate Construction X.1.7 to define the q-de Rham complex $q\Omega^*_{D,\square}$ as promised. For future use, we observe that the above construction is functorial in the triple $(P, J, \{x_1, ..., x_n\} \subset P)$.

With Construction 2.7 in hand, we can prove the promised theorem:

Theorem 2.9 (Comparison with q-de Rham cohomology)**.** *Let $(R,\square)$ be a framed $\mathbf{Z}_p$-algebra. Then there is a natural identification*

$$q\Omega_R \simeq q\Omega^*_{R,\square}$$

in $\mathcal{D}_{comp}(A)$.

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[4]The notation $q\Omega^*_{D,\square}$ is inconsistent with that of Construction X.1.7, but we hope that there is no confusion.

Proof sketch. This argument is analogous to the proof of the crystalline-de Rham comparison in [1]. The framing $\square : \mathbf{Z}_p[x_1, ..., x_n]^\wedge \to R$ deforms uniquely to a formally étale map $A[x_1, ..., x_n]^\wedge \to P$ of δ-A-algebras as in Exercise X.1.6. Applying Construction 2.2 using as input the surjection $P \to R$, we obtain a cosimplicial δ-ring $P^\bullet$, a cosimplicial ideal $J^\bullet \subset P^\bullet$ with $P^\bullet/J^\bullet \simeq R$, and a cosimplicial q-PD envelope $D_{J^\bullet,q}(P^\bullet)$ that computes $q\Omega_R$ (by Remark 2.4). As the framing for $P = P^0$ defines a framing for each term P^n of the cosimplicial ring $P^\bullet$ compatibly with the transition maps, we can apply Construction 2.7 to the diagram $D_{J^\bullet,q}(P^\bullet)$ to obtain a cosimplicial chain complex $M^{\bullet,*} := q\Omega^*_{D_{J^\bullet,q}(P^\bullet),\square}$. We shall compute the totalization of $M^{\bullet,*}$ in two different ways to obtain the desired quasi-isomorphism.

First, the rows $M^{\bullet,i}$ are acyclic for $i > 0$ by the same argument as in [1, Lemma 2.15], so the totalization of the bicomplex $M^{\bullet,*}$ is quasi-isomorphic to that of its 0-th row $D_{J^\bullet,q}(P^\bullet)$. As the latter computes $q\Omega_R$ by Cech-Alexander theory, it follows that $M^{\bullet,*}$ totalizes to $q\Omega_R$.

On the other hand, all the cosimplicial transition maps $M^{i,*} \to M^{j,*}$ are quasi-isomorphisms: by derived Nakayama, this can be checked modulo $(q-1)$, where it follows from the Poincaré lemma (and Proposition 1.6 to identify the result of reducing modulo $q-1$), as in [1, Lemma 2.13]. It follows that the totalization of the cosimplicial complex $M^{\bullet,*}$ is quasi-isomorphic to its 0-th column, which is the q-de Rham complex $q\Omega^*_{R,\square}$.

Combining the previous two paragraphs gives the desired quasi-isomorphism. ∎

Remark 2.10 (Frobenius is an isogeny)**.** Let R be a formally smooth $\mathbf{Z}_p$-algebra. The q-crystalline cohomology complex $q\Omega_R$ comes equipped with a Frobenius endomorphism ϕ_R that is linear over the Frobenius on A. If R comes equipped with a framing $\square$, then we may use Theorem 2.9 to transport this endomorphism to the q-de Rham complex $q\Omega^*_{R,\square}$, at least in the derived category. In fact, it translates to an explicit endomorphism that we describe explicitly next.

Assume for notational simplicity that $\dim(R/p) = 1$, so $\square$ is an étale map $\mathbf{Z}_p[x]^\wedge \to R$. The q-de Rham complex is then given by

$$q\Omega^*_{R,\square} := \left(R[\![q-1]\!] \xrightarrow{\nabla_{q,x}} R[\![q-1]\!]dx\right)$$

as in Construction X.1.7. The Frobenius endomorphism ϕ_R of $q\Omega_R$ is given by the endomorphism of the above complex that is $f \mapsto \phi(f)$ in degree 0 and $gdx \mapsto \phi(g)x^{p-1}[p]_q dx$ in degree 1. (Here ϕ denotes the Frobenius endomorphism of the δ-A-algebra $R[\![q-1]\!]$, and is determined uniquely by requiring $\delta(x) = \delta(q) = 0$.)

One can use this description to show that the linearized Frobenius $\phi_R : \phi_A^* q\Omega_R \to q\Omega_R$ is a $[p]_q$-isogeny, i..e, admits an inverse up to multiplication by $[p]_q^d$, where $d = \dim(R/p)$. In fact, one can also prove a finer statement involving the Berthelot-Ogus $L\eta_{[p]_q}$-functor in this manner (see [2, Proposition 9.17] for a variant). Using Theorem 2.5 and a deformation argument, similar comments apply to prismatic cohomology as well.

Remark 2.11 (The relative case)**.** In this lecture, we have only explained q-crystalline cohomology for formally smooth $\mathbf{Z}_p$-algebras. More generally, given any q-PD pair (D, I) and a formally smooth D/I-algebra R, one can define a q-crystalline site $(R/D)_{\mathrm{qcrys}}$ and the results discussed above concerning q-crystalline cohomology have natural variants in this setting. One may use this to relate the $A\Omega$-theory from [2] to the prismatic cohomology discussed in these lectures.

References

[1] B. Bhatt, A. J. de Jong, *Crystalline cohomology and de Rham cohomology*
[2] B. Bhatt, M. Morrow, P. Scholze, *Integral p-adic Hodge theory.*
[3] J. Pridham, *On q-de Rham cohomology via Λ-rings*
[4] P. Scholze, *Canonical q-deformations in arithmetic geometry*
[5] The Stacks Project