LECTURE X: THE q-DE RHAM COMPLEX

In this lecture, we formally introduce the q-de Rham complex and formulate (a variant of) Scholze's co-ordinate independence conjecture from [4]; this lecture is essentially an exposition of some portions of [4], and we encourage the reader to consult [4] for more, including references to earlier work on q-de Rham cohomology.

The goal of the remaining lectures is twofold. First, we will prove this conjecture by realizing q-de Rham complexes via *q-crystalline cohomology*. Secondly, we shall relate q-crystalline cohomology to prismatic cohomology. Combining the two gives explicit complexes computing prismatic cohomology, in much the same way that the de Rham complex of a $\mathbf{Z}_p$-lift computes the crystalline cohomology of a smooth $\mathbf{F}_p$-algebra.

1. The q-de Rham complex

Notation 1.1. We work over the ring $\mathbf{Z}_p[\![q-1]\!]$ of formal power series over $\mathbf{Z}_p$. Unless otherwise specified, regard $\mathbf{Z}_p$ as a $\mathbf{Z}_p[\![q-1]\!]$-algebra by sending q to 1. Let $[n]_q := \frac{q^n-1}{q-1} = \sum_{i=0}^{n-1} q^i \in \mathbf{Z}_p[\![q-1]\!]$ be the q-analog of n, so $[n]_q \mapsto n$ under $\mathbf{Z}_p[\![q-1]\!] \xrightarrow{q\mapsto 1} \mathbf{Z}_p$. We write $\mathbf{Z}_p[\epsilon_p] := \mathbf{Z}_p[\![q-1]\!]/([p]_q)$, where $q = \epsilon_p$ is a primitive p-th root of 1.

The fundamental construction is the following (see also Remark I.3.2 (f)):

Construction 1.2 (The q-de Rham complex of a polynomial ring, following Aomoto-Jackson)**.** Let $R = \mathbf{Z}_p[x]^\wedge$ be the p-adic completion of the polynomial ring. Define its q-de Rham complex as the 2-term complex

$$q\Omega^*_{R,\square} := \left(R[\![q-1]\!] \xrightarrow{\nabla_q} R[\![q-1]\!]dx \right)$$

where $R[\![q-1]\!]$ is the $(q-1)$-adic completion of $R[q]$, the term dx denotes a formal label, and the differential is given by the q-derivative ∇_q defined as

$$\nabla_q(f(x)) = \frac{f(qx)-f(x)}{qx-x}dx. \tag{1}$$

(The $\square$ stands for "framing" and indicates the co-ordinate dependence of the construction.) Note this formula makes sense because $f(qx) \equiv qx \mod (qx-x)$. Also, we have

$$\nabla_q(x^n) = \frac{q^n x^n - x^n}{qx-x}dx = [n]_q x^{n-1}dx.$$

Using this formula, it is not difficult to see that $q\Omega^*_{R,\square}$ is a deformation of the (p-adically completed) de Rham complex $\Omega^*_{R/\mathbf{Z}_p}$ along the map $\mathbf{Z}_p[\![q-1]\!] \xrightarrow{q\mapsto 1} \mathbf{Z}_p$, i.e., there is an isomorphism

$$q\Omega^*_{R,\square}/(q-1) \simeq \Omega^*_{R/\mathbf{Z}_p}.$$

More generally, one can write down an analogous definition of $q\Omega^*_{R,\square}$ for $R := \mathbf{Z}_p[x_1, ..., x_n]^\wedge$ (see Construction 1.7 for a more general construction).

Remark 1.3 (The q-deformation is not constant)**.** Construction 1.2 produces an object that genuinely differs from the de Rham complex, even in the derived category, i.e., the q-de Rham complex $q\Omega^*_{R,\square}$ is not quasi-isomorphic to the constant deformation $\Omega^*_{R/\mathbf{Z}_p}\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{Z}_p[\![q-1]\!]$. An easy way to see this is to compute cohomology groups (argument omitted). A slightly more conceptual argument involves observing that the two look quite different when reduced modulo $[p]_q$. Indeed,

$$\big(\Omega^*_{R/\mathbf{Z}_p}\widehat{\otimes}_{\mathbf{Z}_p}\mathbf{Z}_p[\![q-1]\!]\big)/[p]_q \simeq \Omega^*_{R/\mathbf{Z}_p}\otimes_{\mathbf{Z}_p}\mathbf{Z}_p[\epsilon_p]$$

has H^0 given by the constants $\mathbf{Z}_p[\epsilon_p]$. On the other hand, $H^0(q\Omega^*_{R,\square}/[p]_q)$ contains not only the constants $\mathbf{Z}_p[\epsilon_p]$, but also all classes coming from the monomials x^i whose degree i is divisible by p: this is because $[p]_q \mid [i]_q$ if $p \mid i$. Remark 1.14 gives a more conceptual description of this specialization.

Note that Construction 1.2 is sensitive to the choice of the co-ordinate x on $R = \mathbf{Z}_p[x]^\wedge$ and thus not functorial in any obvious way: it is not even clear how $x \mapsto x+1$ would act on $q\Omega^*_{R,\square}$. We shall later see that there is a well-defined action in the derived category. First, we comment on the multiplicative structure of the q-de Rham complex.

Remark 1.4 (Multiplicative structure)**.** In Construction 1.2, the q-derivative ∇_q satisfies the following q-Leibnitz rule

$$\nabla_q(f(x)g(x)) = f(x)\nabla_q(g(x)) + g(qx)\nabla_q(f(x)). \tag{2}$$

One can use this to promote $q\Omega_{R,\square}$ naturally to a differential graded algebra compatible with the ring structure on $q\Omega^0_{R,\square}$. To do so, one must endow $q\Omega^1_{R,\square}$ with a $q\Omega^0_{R,\square}$-bimodule structure (corresponding to the left and right multiplication of the algebra on itself) such that we also have

$$\nabla_q(f(x)g(x)) = f(x)\cdot_L \nabla_q(g(x)) + \nabla_q(f(x))\cdot_R g(x),$$

where $\cdot_L$ and $\cdot_R$ denote the left and right actions. This can be accomplished as follows: define the left action to be the standard one, and the right action to be given by

$$b(x)dx\cdot_R a(x) = a(qx)b(x)dx.$$

With these definitions, $q\Omega^*_{R,\square}$ is naturally a $\mathbf{Z}_p[\![q-1]\!]$-dga. Note that since the left and right actions are distinct, this construction does *not* endow $q\Omega^*_{R,\square}$ with the structure of a commutative $\mathbf{Z}_p[\![q-1]\!]$-dga. We shall see later that $q\Omega^*_{R,\square}$ is actually commutative up to homotopy (in fact, it is naturally an E_∞-$\mathbf{Z}_p[\![q-1]\!]$-algebra).

We now explain how to extend the construction of the q-de Rham complex to any formally smooth $\mathbf{Z}_p$-algbera equipped with étale co-ordinates. The following definition is convenient:

Definition 1.5 (Framings)**.** Let S be a formally smooth[1] $\mathbf{Z}_p$-algebra. A *framing* for S is a formally étale map $\square : \mathbf{Z}_p[x_1,...,x_n]^\wedge \to S$; we call the pair $(S,\square)$ a *framed pair*.

We shall use framings to deform across $\mathbf{Z}_p[\![q-1]\!] \xrightarrow{q\mapsto 1} \mathbf{Z}_p$ using the following exercise.

Exercise 1.6 (Deforming framings)**.** Fix a framing $\square : \mathbf{Z}_p[x_1,...,x_n]^\wedge \to S$ of a formally smooth $\mathbf{Z}_p$-algebra S. Let $S[\![q-1]\!]$ be the $(q-1)$-adic completion of $S[q]$.

(1) The framing map $\square$ deforms uniquely to a $(p,q-1)$-completely étale map

$$\widetilde{\square} : \mathbf{Z}_p[q-1,x_1,...,x_n]^\wedge_{(p,q-1)} \to S[\![q-1]\!]$$

of formally smooth $\mathbf{Z}_p[\![q-1]\!]$-algebras. This deformed map (and thus also $\square$) is flat.

(2) The ring $S[\![q-1]\!]$ is flat and topologically free over $\mathbf{Z}_p[\![q-1]\!]$.

For the flatness assertions, we refer to [2, Proposition 5.1].

[1]We recall again our convention for what this means: S is p-adically complete and p-torsionfree, and S/p is a smooth $\mathbf{F}_p$-algebra.

We now explain how to extend Construction 1.2 to arbitrary framed pairs (with Construction 1.2 corresponding to $\square = \mathrm{id}$). The strategy is to interpret the q-derivative (1) for the polynomial algebra in Construction 1.2 in terms of the infinitesimal automorphism $f(x) \mapsto f(qx)$ of the polynomial ring; we then use the formal étaleness of the framing map to extend this construction to more general framed pairs.

Construction 1.7 (Extending to framed smooth $\mathbf{Z}$-algebras)**.** Fix a framed formally smooth $\mathbf{Z}_p$-algebra $(S, \square)$. Consider the (unique) deformation

$$\widetilde{\square} : \mathbf{Z}_p[q-1, x_1, ..., x_n]^{\wedge}_{(p,q-1)} \to S[\![q-1]\!]$$

of the framing $\square$ coming from Exercise 1.6. For each $i \in \{1, ..., n\}$, let γ_i denote the automorphism of $\mathbf{Z}_p[q-1, x_1, ..., x_n]^{\wedge}_{(p,q-1)}$ given by scaling x_i by q, i.e., $\gamma_i(x_i) = qx_i$ and $\gamma_i(x_j) = x_j$ for $j \neq i$. This automorphism is congruent to 1 modulo the topologically nilpotent ideal $(qx_i - x_i)$, and thus lifts uniquely to an automorphism of $S[\![q-1]\!]$ that is also congruent to the identity modulo $(qx_i - x_i)$; by abuse of notation, we also call this automorphism γ_i. We then define the q-derivative $\nabla_{q,i} : S[\![q-1]\!] \to S[\![q-1]\!]dx_i$ to be the map

$$\nabla_{q,i}(f) = \frac{\gamma_i(f) - f}{qx_i - x_i} dx_i \in S[\![q-1]\!]dx_i, \tag{3}$$

where the fraction makes sense as $\gamma_i \equiv \mathrm{id} \mod (qx_i - x_i)$ and $qx_i - x_i$ is a nonzerodivisor in $S[\![q-1]\!]$. It is easy to then put these together for $i \in \{1, 2, ..., n\}$ to build the q-de Rham complex $q\Omega^*_{S,\square}$. In fact, if one ignores the formal label dx_i, then $\{\nabla_{q,i}\}_{i=1,...,n}$ can be regarded as a family of commuting endomorphisms of the $\mathbf{Z}_p[\![q-1]\!]$-module $S[\![q-1]\!]$; the q-de Rham complex $q\Omega^*_{S,\square}$ is then simply the Koszul complex $\mathrm{Kos}(S[\![q-1]\!]; \nabla_{q,1}, ..., \nabla_{q,n})$ (suitably normalized).

The following lemma is now obligatory:

Lemma 1.8 (The q-de Rham complex is a q-deformation)**.** *Let $(S, \square)$ be a framed pair. Then the q-de Rham complex $q\Omega^*_{S,\square}$ is a q-deformation of the de Rham complex, i.e., there is a natural identification $q\Omega^*_{S,\square}/(q-1) \simeq \Omega^*_{S/\mathbf{Z}_p}$ of complexes.*

Proof. We follow the notation of Construction 1.7. It suffices to show that each q-derivative $\nabla_{q,i}$ reduces modulo $(q-1)$ to the the usual derivative $\frac{\partial}{\partial x_i}$. We shall check the slightly stronger statement that

$$\nabla_{q,i}(f) \equiv \frac{\partial f}{\partial x_i} \mod (qx_i - x_i)$$

for all $f \in S[\![q-1]\!]$ (where we ignore the label dx_i on the left). Using the definition (3) of $\nabla_{q,i}$ in terms of γ_i, this amounts to checking that

$$\gamma_i(f) - f = (qx_i - x_i)\frac{\partial f}{\partial x_i} \mod (qx_i - x_i)^2$$

for all $f \in S[\![q-1]\!]$. As γ_i is congruent to the identity modulo $(qx_i - x_i)$, the induced map

$$\gamma_i - \mathrm{id} : S[\![q-1]\!]/(qx_i - x_i) \to (qx_i - x_i)/(qx_i - x_i)^2.$$

is a derivation: it corresponds to the infinitesimal automorphism γ_i of the square zero extension $S[\![q-1]\!]/(qx_i - x_i)^2 \to S[\![q-1]\!]/(qx_i - x_i)$ under the usual dictionary between derivations and infinitesimal automorphisms. We must show that this derivation is given by $(qx_i - x_i)\frac{\partial}{\partial x_i}$. But this follows from formal étaleness of $\square$ and the analogous statement for $\mathbf{Z}_p[q-1, x_1, ..., x_n]^{\wedge}_{(p,q-1)}$. $\square$

The following conjecture from [4, Conjecture 3.1] postulates that the above construction is coordinate independent if one passes to the derived category.

Conjecture 1.9 (Scholze)**.** *There is a symmetric monoidal functor $S \mapsto q\Omega_S$ from the category of formally smooth $\mathbf{Z}_p$-algebras to $\mathcal{D}_{comp}(\mathbf{Z}_p[\![q-1]\!])$ together with isomorphisms $q\Omega_S \simeq q\Omega^*_{S,\square}$ (in the derived category) for each framed pair $(S,\square)$.*

In particular[2]*, the q-de Rham complex $q\Omega^*_{S,\square}$ of any framed pair $(S,\square)$ is naturally an E_∞-algebra (and thus gives a commutative algebra object in $D_{comp}(\mathbf{Z}_p[\![q-1]\!])$).*

In the next lecture, we shall explain a proof of this conjecture. The strategy will be to introduce a "q-crystalline site" $(S/\mathbf{Z}_p[\![q-1]\!])_{\mathrm{qcrys}}$ (whose definition involves δ-rings) and show that q-de Rham cohomology can be computed by q-crystalline cohomology, i.e., the cohomology of the structure sheaf on this site. It will also be the case that q-crystalline cohomology of S identifies with prismatic cohomology of a certain twist $S^{(1)}$; this provides a concrete representative for the latter (via q-de Rham complexes) and new results (such as the Hodge-Tate and étale comparisons) for the former.

Remark 1.10 (The origin of Conjecture 1.9)**.** Conjecture 1.9 is closely related to results from [1] which implicitly prove the conjecture after base change along the "perfection" map

$$\mathbf{Z}_p[\![q-1]\!] \to A := \mathbf{Z}_p[q^{1/p^\infty}]^\wedge_{(p,q-1)}.$$

To explain this, consider the complete nonarchimedean field $\mathbf{Q}_p^{cyc} = \mathbf{Q}_p(\mu_{p^\infty})^\wedge$. The ring of integers $\mathcal{O}$ of $\mathbf{Q}_p^{cyc}$ is isomorphic to $A/([p]_q)$, where q maps to a primitive p-th root of 1. In particular, this ring is perfectoid with $A_{\mathrm{inf}}(\mathcal{O}) \simeq A$. The paper [1] constructs a symmetric monoidal functor $R \mapsto A\Omega_R$ on formally smooth $\mathcal{O}$-algebras that is valued in $\mathcal{D}_{comp}(A)$. Moreover, a calculation (see [1, §9]) shows that if one fixes a formally étale map $\mathcal{O}[x_1,...,x_n]^\wedge \to R$ with x_i invertible in R, then $A\Omega_R$ is computed by (a variant over A of) a q-de Rham complex. In particular, the resulting functor $S \mapsto A\Omega_{S\widehat{\otimes}_{\mathbf{Z}_p}\mathcal{O}}$ on formally smooth $\mathbf{Z}_p$-algebras solves Conjecture 1.9 after base change along $\mathbf{Z}_p[\![q-1]\!] \to A$, as asserted.

Remark 1.11 (Conjecture 1.9 after rationalization)**.** The q-derivative ∇_q from (1) can be related to the usual derivative by a simple formula if one allows denominators. In particular, one can use this to prove Conjecture 1.9 after completed base change along $\mathbf{Z}_p[\![q-1]\!] \to \mathbf{Q}_p[\![q-1]\!]$; in fact, one shows that the q-de Rham complex becomes isomorphic to the usual de Rham complex after this base change. We refer to [4, Lemma 4.1] as well as [1, Lemma 12.4] for more.

Remark 1.12 (Other work)**.** Prior to the work described in these lectures, Pridham [3] made progress on Conjecture 1.9: (very roughly speaking) he explained why the q-de Rham complex $q\Omega^*_{S,\square}$ of a framed pair $(S,\square)$ depends only on the δ-structure of S determined by the framing (Lemma II.2.9). Progress has also been announced by Masullo.

Remark 1.13 (Conjecture 1.9 without p-adically completing)**.** In these lectures, we have restricted attention to the local case, i.e., we work with p-adically complete rings for a fixed prime p. This restriction is not necessary for Constructions 1.2 or 1.7: the constructions make perfect sense for étale $\mathbf{Z}[x_1,...,x_n]$-algebras S without any p-adic completions[3]. In fact, Scholze's [4, Conjecture 3.1] is also formulated in the global setting, and Pridham's work [3] mentioned in Remark 1.12 is also partially in the global setting. However, there is a relatively formal procedure to patch together the local theory with the rational result mentioned in Remark 1.11 to obtain the global independence. For this reason, we stick to the p-adically complete setting in these lectures.

[2]The symmetric monoidal structure on $S \mapsto q\Omega_S$ ensures that the multiplication map on S defines a multiplication on $q\Omega_S$. A more careful variant of this observation allows us to pass the commutative algebra structure on S to obtain a commutative algebra structure on $q\Omega_S \in \mathcal{D}_{comp}(\mathbf{Z}_p[\![q-1]\!])$. As $\mathcal{D}_{comp}(\mathbf{Z}_p[\![q-1]\!])$ is an ∞-category, a commutative algebra structure on an object is fairly elaborate; for example, any representative complex A^* of such an object acquires the structure of an E_∞-$\mathbf{Z}_p[\![q-1]\!]$-algebra.

[3]But the $(q-1)$-adic completion is necessary for Construction 1.7.

Remark 1.14 (Roots of unity)**.** A commonly observed phenomenon when working with q-analogs is that the behaviour when q is a non-trivial root of unity often mimics characteristic p behaviour. A similar phenomenon holds true for the q-de Rham complex: for a framed algebra $(S, \square)$, one has "Cartier isomorphisms"

$$H^i(q\Omega^*_{S,\square} \otimes_{\mathbf{Z}_p[\![q-1]\!]} \mathbf{Z}_p[\epsilon_p]) \simeq \Omega^i_{R/\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\epsilon_p].$$

These have been constructed directly by Scholze [4, Proposition 3.4] by a procedure reminiscent of Construction V.5.8, and will follow later from the Hodge-Tate comparison for prismatic cohomology (Theorem V.3.8) once we relate q-de Rham complexes to prismatic cohomology.

Remark 1.15 (Globalization and étale comparison)**.** Follow the notation in Conjecture 1.9. Assuming the isomorphism in Conjecture 1.9 is independent of the framing modulo $(q-1)$ (which one certainly wants), it follows that the assignment $S \mapsto q\Omega_S$ globalizes to give a complex $q\Omega_{\mathfrak{X}} \in D(\mathfrak{X}, \mathbf{Z}_p[\![q-1]\!])$ of sheaves on any formally smooth $\mathbf{Z}_p$-scheme $\mathfrak{X}$. Motivated by the étale comparison theorem for the $A\Omega$-theory discussed in Remark 1.10, Scholze conjectured [4, Conjecture 3.3] that $H^*(\mathfrak{X}, q\Omega_{\mathfrak{X}})[\frac{1}{q-1}]$ is essentially (up to change of scalars and non-canonically) the p-adic étale cohomology of the geometric generic fibre $\mathfrak{X}_{\mathbf{C}_p}$ for any proper smooth (formal) $\mathbf{Z}_p$-scheme $\mathfrak{X}$; in our approach, this will follow from the relation between q-de Rham and prismatic cohomology combined with the étale comparison theorem for prismatic cohomology (Theorem IX.0.1).

Remark 1.16 (Base rings beyond $\mathbf{Z}_p$)**.** In Construction 1.7, we worked over the base ring $\mathbf{Z}_p$. This was not necessary for the construction: it makes sense over any p-adically complete base ring. However, one does not expect Conjecture 1.9 to hold true over arbitrary base rings. In fact, it fails over the base ring $\mathbf{F}_p$ itself. To see this, note if the analog of Conjecture 1.9 held true in this setting, one would also expect the following (building on Remark 1.15): for any proper smooth (formal) $\mathbf{Z}_p$-scheme $\mathfrak{X}$, one can compute $H^i(\mathfrak{X}, q\Omega_{\mathfrak{X}}/p)$ in terms of the special fibre $\mathfrak{X}_{\mathbf{F}_p}$. Combining with the étale comparison theorem in Remark 1.15, this would imply that the $\mathbf{F}_p$-étale cohomology of the geometric generic fibre $\mathfrak{X}_{\mathbf{C}_p}$ is determined by $\mathfrak{X}_{\mathbf{F}_p}$. However, this latter statement is simply false: there exist smooth proper surfaces over $\mathbf{Z}_2$ with isomorphic special fibres but different $\mathbf{F}_2$-étale cohomologies for the generic fibres (see [1, Remark 2.4]).

References

[1] B. Bhatt, M. Morrow, P. Scholze, *Integral p-adic Hodge theory.*
[2] B. Bhatt, *On the direct summand conjecture and its derived variant*
[3] J. Pridham, *On q-de Rham cohomology via Λ-rings*
[4] P. Scholze, *Canonical q-deformations in arithmetic geometry*