LECTURE I: OVERVIEW

The goal of these lectures is to define prismatic cohomology following [3], and to explain why it unifies various (integral) cohomology theories of interest in p-adic geometry. In this first lecture, we survey some of the global implications of this theory, and a give a flavor of the local results (but there are no definitions).

1. Motivation

Our starting point is the following classical result in complex geometry.

Theorem 1.1 (de Rham, Serre)**.** *Let X be a compact complex manifold. Then there is a natural identification*

$$H^i(X, \mathbf{C}) \simeq H^i_{dR}(X)$$

of $\mathbf{C}$-vector spaces, where the RHS indicates the cohomology of the holomorphic de Rham complex of X.

Explicitly, the dual identification $H_i(X, \mathbf{C}) \simeq H^i_{dR}(X)^\vee$ is given by

$$\gamma \in H_i(X, \mathbf{C}) \mapsto (\omega \mapsto \int_\gamma \omega).$$

Theorem 1.1 thus admits a geometric formulation:

($*$) A non-trivial cycle $\gamma \in H_i(X, \mathbf{C})$ defines a non-trivial obstruction to integrating a differential i-form ω on X: the integral $\int_\gamma \omega$ must vanish.

The two cohomology theories showing up in Theorem 1.1 have very different origins as the singular cohomology groups $H^i(X, \mathbf{C})$ are defined using only the topological space underlying X, while the de Rham cohomology groups $H^i_{dR}(X)$ are defined in terms of the complex geometry of X. The resulting relationship between the topology and the geometry of X coming from Theorem 1.1 is quite close. For example, thanks to Theorem 1.1, one can immediately read off the Betti numbers of X from its de Rham cohomology; this is useful because de Rham cohomology is often easier to compute by sheaf theory. Conversely, one can use Theorem 1.1 to show that certain holomorphic invariants of X are topological in nature, i.e., insensitive to perturbations of the complex structure of X. This interaction between the geometry and topology of complex manifolds can be taken much further, and forms the subject of Hodge theory.

Unfortunately, other than some important low dimensional special cases, classical Hodge theory essentially ignores the "torsion" aspects of the story. The main reason is that integrals over torsion homology classes always vanish, so there is not geometric meaning attached to torsion classes in $H_i(X, \mathbf{Z})$ by Theorem 1.1. One goal of this course is to discuss a partial solution to this problem when X is an algebraic variety. Roughly speaking, we shall show the following variant of ($*$) (which is formulated for simplicity with $\mathbf{Z}/p$-coefficients):

($**$) For any prime p, a non-trivial class in $H_i(X, \mathbf{Z}/p)$ produces a non-trivial obstruction to integrating differential i-forms on "the mod p reduction of X".

To make sense of the phrase "the mod p reduction of X", the coefficients of the equations defining X must lie in a ring where reduction modulo p makes sense. The right context to study such objects is provided by p-adic geometry, and indeed assertion ($**$) properly belongs to the subject of p-adic Hodge theory. Consequently, the main players of this course will be p-adically complete commutative rings, as these arise as the rings of functions on the spaces of interest.

2. Global statements

Let us formulate (∗∗) more precisely. To avoid introducing too much notation, we stick to algebraic varieties with $\mathbf{Q}$-coefficients.

Notation 2.1. Let X be a smooth projective variety over $\mathbf{Q}$ with good reduction outside some integer $N > 0$. Concretely, $X \subset \mathbf{P}^n$ is the subvariety defined by homogeneous polynomials $f_1, ..., f_r$ in $\mathbf{Q}[x_0, ..., x_n]$ such that the coefficients of the f_i's lie in $\mathbf{Z}[1/N] \subset \mathbf{Q}$ and the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$ has rank $n + 1 - \dim(X)$ modulo p for any prime p that does not divide N. For any such prime p, write X_p for the smooth projective variety over $\mathbf{F}_p$ obtained by reducing the equations defining X modulo p, and write X^{an} for the projective complex manifold defined by X.

The precise form (∗∗) is the following theorem from [1].

Theorem 2.2. *For any prime p not dividing N, we have an inequality*

$$\dim H^i(X^{an}, \mathbf{F}_p) \leq \dim H^i_{dR}(X_p), \tag{1}$$

where the RHS indicates the hypercohomology of the algebraic de Rham complex of X_p.

Remark 2.3. Let us make some comments on the limits of Theorem 2.2.

(1) Theorem 2.2 was stated for simplicity for X's that were defined over $\mathbf{Q}$ and had good reduction at p. Inequality (1) holds true more generally for any proper smooth formal scheme $\mathfrak{X}$ over a p-adic valuation ring V (provided one uses étale cohomology to replace the singular cohomology groups). The additional generality of allowing formal schemes is useful in practice as it has applications to rigid-analytic geometry, and is somewhat analogous to formulating Theorem 1.1 for all (i.e., not necessarily algebraic) complex manifolds.

(2) Inequality (1) can be strict. The basic reason for this failure is that "reduction modulo p" is a lossy operation: there can exist pair of varieties X and Y as above such that $X_p \simeq Y_p$, and yet X^{an} and Y^{an} are not isomorphic (or even homotopy-equivalent). Explicitly, one can find [1, §2.1] a threefold X that admits a non-split elliptic fibration structure over an Enriques surface in such a way that the elliptic fibration structure has good reduction at $p = 2$ and splits modulo 2; one then chooses Y to be the corresponding split fibration to obtain an example where $X_p \simeq Y_p$, $H^1(X^{an}, \mathbf{F}_p) = \mathbf{F}_p^2$, and $H^1(Y^{an}, \mathbf{F}_p) = \mathbf{F}_p^3$. In particular, Theorem 2.2 is not as strong as Theorem 1.1.

(3) Inequality (1) cannot be upgraded to a naturally defined subquotient relationship. More precisely, there is a version of Theorem 2.2 with $\mathbf{Z}/p^n$-coefficients for any $n > 0$ with dimension replaced by the length (as a $\mathbf{Z}/p^n$-module) and de Rham cohomology of X_p replaced by its crystalline cohomology. One can then exhibit examples (at least over a ramified valuation ring [1, §2.2]), where the modulo p^2 singular cohomology is isomorphic to $\mathbf{Z}/p^2 \oplus \mathbf{Z}/p^2$, while the corresponding crystalline cohomology module is $\mathbf{Z}/p^2 \oplus \mathbf{Z}/p \oplus \mathbf{Z}/p$. In particular, the former is not a subquotient of the latter.

(4) As explained above, there is a variant of Theorem 2.2 for any proper smooth formal scheme over a p-adic valuation ring V. If V is discrete with absolute ramification index e, and $ie < p - 1$, then the inequality in (1) is actually an equality by previous work of Caruso and Faltings [4, 6].

(5) The hypothesis that X have good reduction at p can be weakened to the assumption that X has semistable reduction at p provided one uses log de Rham cohomology on the RHS of (1); this is the work of Cesnavicus and Koshikawa [5]. Conjecturally, every smooth proper variety X over a p-adic field has this property.

As algebraic de Rham cohomology is directly computable in terms of the defining equations, Theorem 2.2 gives an algebraic approach to detecting the presence of p-torsion in the cohomology of X^{an}. In particular, by the universal coefficients theorem, it has the following consequence:

Corollary 2.4. *Fix a prime p not dividing N. If $\dim H^i_{dR}(X_p) = \dim H^i_{dR}(X^{an})$, then $H^{i+1}(X^{an}, \mathbf{Z})$ has no p-torsion.*

Reading Theorem 2.2 in the converse direction gives a topological explanation for why algebraic de Rham cohomology in characteristic p might exhibit "pathology", i.e., be larger than expected. This is best captured in the following example.

Example 2.5. Assume that 2 does not divide N, and that X^{an} is an Enriques surface (i.e., $\pi_1(X^{an}) \simeq \mathbf{Z}/2$, and the universal cover of X is a surface with trivial canonical bundle). Then $H^1(X^{an}, \mathbf{F}_2) \simeq \mathrm{Hom}(\pi_1(X^{an}, \mathbf{F}_2) \simeq \mathbf{F}_2$, so $H^1_{dR}(X_2) \neq 0$ by Theorem 2.2. As every Enriques surface Y over $\mathbf{F}_2$ has the form X_2 for some X (by Ekedahl, possibly after enlarging the ground field), it follows that $H^1_{dR}(Y) \neq 0$ for any Enriques surface in characteristic 2. This calculation was previously carried out by W. Lang and Illusie [8, 7] using the Bombieri-Mumford classification.

Theorem 1.1 provides an isomorphism of cohomology groups. In contrast, Theorem 2.2 gives merely an inequality of dimensions, and cannot be proven by exhibiting an isomorphism (or even a subquotient relationship) of cohomology groups (Remark 2.3 (2)). Instead, Theorem 2.2 is proven, roughly, by deforming $H^*_{dR}(X_p)$ to $H^*(X^{an}, \mathbf{F}_p)$. More precisely, let $A := \mathbf{F}_p[\![u]\!]$ denote the ring of formal power series of $\mathbf{F}_p$. The variable u plays the role of a deformation parameter. More precisely, if $\mathcal{X}$ denotes the given integral model of X with good reduction at p, we construct a cohomology theory $H^*_A(\mathcal{X})$ valued in finitely generated A-modules with the following properties:

(1) There exists a (non-canonical) identification $H^i_A(\mathcal{X})[\frac{1}{u}] \simeq H^i(X^{an}, \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathbf{F}_p((u))$.

(2) There exists a natural injective map $H^i_A(\mathcal{X})/u \to H^i_{dR}(X_p)$ with cokernel given by the u-torsion $H^{i+1}_A(\mathcal{X})$.

Theorem 2.2 follows immediately from these two properties by the classification of finitely generated modules over the PID A. Property (2) allows us to regard $H^*_A(\mathcal{X})$ as a deformation of the de Rham cohomology of X_p across the map $A \xrightarrow{u \mapsto 0} \mathbf{F}_p$ provided one works in the derived category. This deformation is called *prismatic cohomology*, and its construction and local study following [3] will form the subject of this course.

3. Local structure of prismatic cohomology

The prismatic cohomology theory mentioned above is constructed as the hypercohomology of a complex of sheaves. To understand a complex of sheaves, we may work locally. We therefore restrict to the affine setting in the rest of this section. Let us fix some notation first.

Notation 3.1. Fix a prime p. Let $A = \mathbf{Z}_p[\![u]\!]$, so A is a noetherian local ring of dimension 2 with residue field $\mathbf{F}_p$. Write $\phi : A \to A$ for the ring homomorphism determined by $u \mapsto u^p$; this map is a lift of the Frobenius endomorphism of A/p. Write $I \subset A$ for the ideal generated by $u - p \in A$, so $A/I \simeq \mathbf{Z}_p$ via $u \mapsto p$. Write $\tilde{\theta} : A \to \mathbf{Z}_p$ for the resulting map[1]. Note that $\mathbf{Z}_p$ carries a unique Frobenius lift (namely, the identity map), but the map $\tilde{\theta}$ is *not* compatible with the Frobenius lifts.

Let R be the p-adic completion of a smooth $\mathbf{Z}_p$-algebra. Write $R_{\mathbf{F}_p} := R/p$ for the special fibre, and let $\Omega^*_{R_{\mathbf{F}_p}/\mathbf{F}_p}$ be its algebraic de Rham complex. Let R_C denote the geometric generic fibre of $R/\mathbf{Z}_p$, i.e., $R_C = R \otimes_{\mathbf{Z}_p} C$ for an algebraic closure C of $\mathbf{Q}_p$. It will also be useful to name $\Omega^*_{R/\mathbf{Z}_p}$,

[1] The triple (A, ϕ, I) will be an example of a *prism*. Everything below can be formulated over any prism.

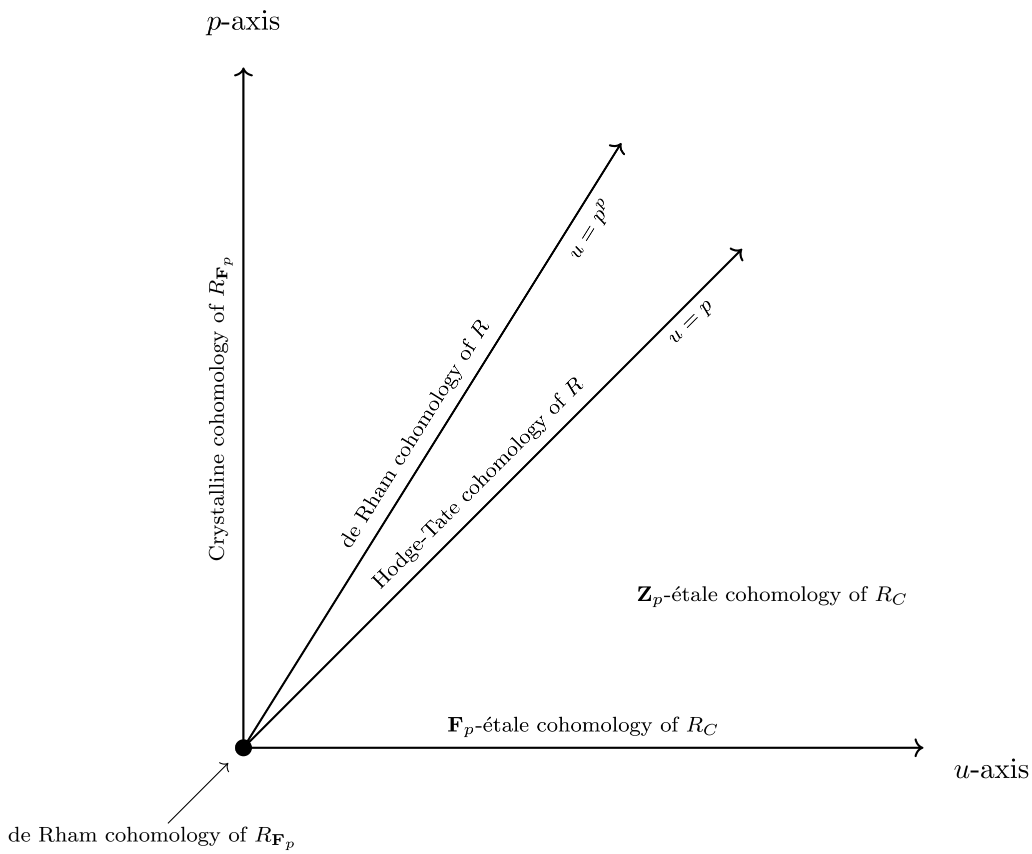

FIGURE 1. The "values" over $\Delta_{R/A}$ over $\mathrm{Spec}(\mathbf{Z}_p[\![u]\!])$ as provided by Theorem 3.3

the continuous de Rham complex of R over $\mathbf{Z}_p$; its mod p reduction identifies with the de Rham complex $\Omega^*_{R_{\mathbf{F}_p}/\mathbf{F}_p}$ of $R_{\mathbf{F}_p}$.

The algebraic de Rham complex of R is computable in terms of the equations defining R. We give one example that will recur throughout this course.

Example 3.2. Say R is the p-adic completion of the Laurent polynomial ring $\mathbf{Z}[x, x^{-1}]$. Then $\Omega^1_{R/\mathbf{Z}_p}$ is a free R-module of rank 1 with basis $\frac{dx}{x}$, and the higher exterior powers vanish. Using the grading by degree of the monomial, and noting that $d(x^i) = ix^i \frac{dx}{x}$, we obtain the following explicit presentation for the de Rham complex

$$\Omega^*_{R/\mathbf{Z}_p} = \widehat{\bigoplus_{i \in \mathbf{Z}}} \Big(\mathbf{Z}_p \cdot x^i \xrightarrow{i} \mathbf{Z}_p \cdot x^i \frac{dx}{x} \Big).$$

A similar presentation can be written for a higher dimensional torus.

Prismatic cohomology provides a cohomology theory attached to R that takes values in finitely generated A-modules with the following features: the "value" of this cohomology theory at the closed point $\mathrm{Spec}(\mathbf{F}_p) \hookrightarrow \mathrm{Spec}(A)$ is de Rham cohomology of $R_{\mathbf{F}_p}$, while its "value" away from the vanishing set $V(I) \subset \mathrm{Spec}(A)$ recovers the étale cohomology of R_C. A pictorial description is given in Figure 1. A precise statement of the theorem is the following.

Theorem 3.3. *To the ring R we can functorially attach a complex $\Delta_{R/A}$ of (p,u)-adically complete A-modules (viewed as an object of the derived category $D(A)$ of A-modules) together with a "Frobenius" map $\phi_{R/A} : \phi^*\Delta_{R/A} \to \Delta_{R/A}$. This pair $(\Delta_{R/A}, \phi_{R/A})$ satisfies the following properties:*

1. *(The shtuka property) The "Frobenius" map $\phi_{R/A}$ is a quasi-isomorphism on inverting $u - p \in A$. In particular, it becomes an isomorphism after base change along the obvious map $A \to \mathbf{F}_p((u))$ showing up in (3) below.*
2. *(de Rham comparison for $R_{\mathbf{F}_p}$) There exists a natural quasi-isomorphism*
$$\phi^*\Delta_{R/A} \otimes^L_A \mathbf{F}_p \simeq \Omega^*_{R_{\mathbf{F}_p}/\mathbf{F}_p},$$
where the map $A \to \mathbf{F}_p$ is determined by $u \mapsto 0$. In other words, the object $\phi^*\Delta_{R/A} \in D(A)$ lifts the de Rham complex of the special fibre $R_{\mathbf{F}_p}$ along the map $A \to \mathbf{F}_p$.
3. *(Étale comparison) There exists a non-canonical quasi-isomorphism*
$$\left(\Delta_{R/A} \otimes^L_A \overline{\mathbf{F}_p((u))}\right)^{\phi_{R/A}\otimes\phi=1} \simeq R\Gamma_{et}(\mathrm{Spec}(R_C), \mathbf{F}_p),$$
where $\overline{\mathbf{F}_p((u))}$ denote an algebraic closure of $\mathbf{F}_p((u))$, and the map $A \to \overline{\mathbf{F}_p((u))}$ is the obvious one. In short, the étale cohomology of R_C can be recovered from $\Delta_{R/A}[\frac{1}{u-p}]$.
4. *(The Hodge-Tate comparison) There exists a natural identification*
$$H^i(\Delta_{R/A} \otimes^L_A \mathbf{Z}_p) \simeq \Omega^i_{R/\mathbf{Z}_p},$$
where the RHS denotes the module of continuous differential i-forms on R, and the implicit map $A \to \mathbf{Z}_p$ is $\tilde{\theta}$. (Here we suppress certain "Breuil-Kisin" twists that are trivialized by the choice of the generator $u - p \in I$.)

Moreover, there are natural variants of (2) and (3) with $\mathbf{Z}/p^n$-coefficients.

Remark 3.4. Let us make a few remarks explaining the relation to the preceding discussion.

(a) Write $\Omega^*_{R/\mathbf{Z}_p}$ for the continuous de Rham complex of R over $\mathbf{Z}_p$, so $\Omega^*_{R_{\mathbf{F}_p}/\mathbf{F}_p}$ is naturally the mod p reduction of $\Omega^*_{R/\mathbf{Z}_p}$. The de Rham comparison isomorphism in Theorem 3.3 (2) can be upgraded to:

(2') (de Rham comparison for R) There exists a canonical quasi-isomorphism
$$\phi^*\Delta_{R/A} \otimes^L_A \mathbf{Z}_p \simeq \Omega^*_{R/\mathbf{Z}_p},$$
where the map $A \to \mathbf{Z}_p$ is $\tilde{\theta}$. In other words, the object $\phi^*\Delta_{R/A} \in D(A)$ lifts the de Rham complex of R along the map $\tilde{\theta} : A \to \mathbf{Z}_p$.

(b) (A mixed characteristic Cartier isomorphism) As the Frobenius map on $\mathbf{F}_p$ is the identity, the base changes $\phi^*\Delta_{R/A} \otimes^L_A \mathbf{F}_p$ and $\Delta_{R/A} \otimes^L_A \mathbf{F}_p$ are naturally identified. The former is computed by Theorem 3.3 (2), while the latter can be computed by Theorem 3.3 (4). Combining these, we obtain a canonical isomorphism
$$H^i(\Omega^*_{R_{\mathbf{F}_p}/\mathbf{F}_p}) \simeq \Omega^i_{R_{\mathbf{F}_p}/\mathbf{F}_p}.$$
This is simply the classical Cartier isomorphism for the smooth $\mathbf{F}_p$-algebra $R_{\mathbf{F}_p}$. One can thus view the Hodge-Tate comparison theorem and $(2')$ above as providing a mixed characteristic lift of the Cartier isomorphism.

(c) (Globalization) The assignment $R \mapsto (\Delta_{R/A}, \phi_{R/A})$ satisfies Zariski descent in R. Concretely, this allows us to define a prismatic complex of sheaves $(\Delta_{\mathcal{X}/A}, \phi_{\mathcal{X}/A})$ for any smooth formal $\mathbf{Z}_p$-scheme $\mathcal{X}$ by glueing together the complexes $\Delta_{R/A}$ for various affine opens $\mathrm{Spf}(R) \subset \mathcal{X}$.

Writing $R\Gamma_A(\mathcal{X})$ for the hypercohomology of $\Delta_{\mathcal{X}/A}$, then the étale comparison isomorphism takes on a more appealing form when $\mathcal{X}$ is proper: there is a non-canonical identification

$$R\Gamma_A(\mathcal{X}) \otimes^L_A \mathbf{F}_p((u)) \simeq R\Gamma(X_C, \mathbf{F}_p) \otimes_{\mathbf{F}_p} \mathbf{F}_p((u))$$

in $D(\mathbf{F}_p((u)))$. Moreover, the cohomology groups of $R\Gamma_A(\mathcal{X}) \otimes^L_A \mathbf{F}_p[\![u]\!]$ provide the cohomology theory mentioned at the end of §2.

(d) In practice, as the prismatic complex $\Delta_{R/A}$ is $(p-u)$-adically complete, the Hodge-Tate comparison isomorphism gives the most effective tool to control prismatic cohomology.

(e) (Commutative algebra structure) Taking wedge products of differential forms endows the de Rham complex with the structure of a commutative differential graded algebra, so its image in the derived category admits a natural commutative algebra structure. Similarly, the prismatic complex $\Delta_{R/A}$ naturally admits the structure of a commutative algebra object in the derived category $D(A)$. (In fact, one can naturally promote it to a commutative algebra structure in the derived ∞-category; this has concrete consequences such as an action of the Steenrod algebra on $H^*(\Delta_{R/A})$.) The map $\phi_{R/A}$ and all the isomorphisms of Theorem 3.3 are multiplicative with respect to this structure and the standard commutative algebra structure on the various cohomology theories. The multiplicative structure on $\Delta_{R/A}$, however, is not as rigid as the one on de Rham cohomology (see (g) below).

(f) (q-de Rham complexes) Consider ring $B = \mathbf{Z}_p[\![q-1]\!]$ equipped with the Frobenius lift $\phi : B \to B$ determined by $\phi(q) = q^p$. Let $J \subset B$ be the ideal generated by $[p]_q := \frac{q^p-1}{q-1}$, so B/J identifies with $\mathbf{Z}_p[\epsilon_p]$. The triple (B, ϕ, J) gives a prism, and one has an analog of Theorem 3.3 for any B/J-algebra R that is the p-adic completion of a smooth B/J-algebra, i.e., there exists a complex $\Delta_{R/B}$ of B-modules equipped with a "Frobenius" map $\phi_{B/A}$ and satisfying natural analogs of (1) through (4). When the ring R is obtained as the base change of a $\mathbf{Z}_p$-algebra to B/J, one can give an explicit description of $\Delta_{R/B}$ as a "q-deformation" of the de Rham complex. Let us explain this when R is the p-adic completion of the Laurent polynomial ring $B/J[x, x^{-1}]$.

To begin, as in Example 3.2, we have a presentation

$$\Omega^*_{R/(B/J)} = \widehat{\bigoplus_{i \in \mathbf{Z}}} \Big(B/J \cdot x^i \xrightarrow{i} B/J \cdot x^i \frac{dx}{x} \Big).$$

The complex $\Delta_{R/B}$ is a lift of the above complex along $B \to B/J$. To describe this lift, we follow the standard convention

$$[i]_q := \frac{q^i - 1}{q-1} = 1 + q + q^2 + ... + q^{i-1} \in B$$

for an integer $i \geq 0$. Note that $[i]_q \mapsto i$ under $B \to B/J$; the polynomial $[i]_q$ is often called the "q-analog of i". With this notation, $\Delta_{R/B}$ turns out to be explicitly (up to quasi-isomorphism) given by

$$\Delta_{R/(B/J)} \simeq \widehat{\bigoplus_{i \in \mathbf{Z}}} \Big(B \cdot x^i \xrightarrow{[i]_q} B \cdot x^i \frac{dx}{x} \Big).$$

Note that reducing the Frobenius pullback $\phi^* \Delta_{R/(B/J)}$ modulo $[p]_q$ indeed gives the de Rham complex since $[i]_{q^p}$ is congruent to i modulo $q^p - 1$, and hence also modulo $[p]_q$. The parenthetical phrase "up to quasi-isomorphism" appearing above is important: the actual complex appearing on the right is incredibly sensitive to the choice of the co-ordinate $x \in B$, and it is not even clear how linear automorphisms of R (such as $x \mapsto x + p$) would act on this complex. Nevertheless, the existence of the preceding quasi-isomorphism ensures that the RHS is functorial in the automorphisms of B, up to quasi-isomorphisms. We refer to [9] for a more thorough discussion of q-de Rham complexes, and simply mention that the prismatic approach resolves the "independence of choices" conjectures in [9].

(g) The preceding example points to an important property of the prismatic complex $\Delta_{R/A}$: even though the de Rham complex $\Omega^*_{R/\mathbf{Z}_p}$ is an actual complex (as opposed to merely being an object of a derived category), the prismatic complex $\Delta_{R/A}$ only lives in the derived category $D(A)$, and does not admit a preferred chain complex representative. In fact, using the Steenrod operations mentioned in (e), one can show that there does not exist a commutative differential graded A-algebra representing the commutative algebra $\Delta_{R/A}$. This makes it difficult to construct $\Delta_{R/A}$ "by hand" like the de Rham complex, and instead one must proceed by finding a suitable universal property.

Remark 3.5 (A comment on the proofs)**.** There are currently 3 distinct constructions of prismatic complexes available at the moment.

In [1], a variant of the complex $\Delta_{R/A}$ (where A was replaced by a much larger *period* ring) was constructed using the Berthelot-Ogus-Deligne $L\eta$-functor and the almost purity theorem.

An alternative construction via topological Hochschild homology is given in [2]. This construction, which requires as input a highly non-trivial calculation from homotopy theory due to Bokstedt, also applies to the ring $A = \mathbf{Z}_p[\![u]\!]$ above; however, it is not known if the topological theory can be adapted to cover every base prism. An important advantage of this approach is that it lays bare the structure of an important filtration on prismatic complexes (the Nygaard filtration), which can then be used to define a meaningful Tate twist in mixed characteristic.

In this course, we shall follow the construction in [3] in terms of the prismatic site (which may be regarded as a mixed characteristic variant of the crystalline site). This construction is simultaneously the most general and the most elementary. In fact, as we shall explain in the course, one can use the prismatic approach to reprove basic results from the theory of perfectoid spaces (such as the almost purity theorem) in an easier fashion.

4. Outline of course

(1) δ-rings (aka rings with a lift of Frobenius), prisms, and perfectoid rings.

(2) The prismatic site and the Hodge-Tate comparison theorem, the q-crystalline site.

(3) Applications to commutative algebra and perfectoid geometry.

(4) The de Rham and étale comparison theorems.

(5) Connections to algebraic topology (time permitting).

The first third of the course shall assume familiarity with commutative algebra. Later on, we shall occasionally use the language of derived categories, and some acquaintance with the notion of a Grothendieck topology will also be useful.

References

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