Problem set 5

Notation 0.1. When talking about coherent sheaves, we use $f^{-1}$ to denote the set-theoretic inverse image functor associated a map $f : X \to Y$ of schemes.

1. The goal of this exercise is prove a theorem of Nagata showing that henselizations of noetherian local rings are also noetherian. Let $(R, m, k)$ be a noetherian local ring.
   
   (a) Show that $R/m^n \simeq R^h/m^n$, and hence $\hat{R} \simeq \hat{R}^h$.
   
   (b) Using the criterion formulated in terms of the base-change behaviour of finitely generated ideals, show that $R^h \to \hat{R}^h$ is faithfully flat.
   
   (c) Show that $R^h$ is noetherian.
   
   (d) (*) Show that $R^{sh}$ is noetherian (or read EGA III, Chapter 0, Lemma 10.3.1.3).

2. (*) Let $(R, m, k)$ be a henselian local ring. Show that reduction modulo $m$ defines an equivalence of categories between finite étale covers of $R$, $\hat{R}$, and $k$.

3. (Topological description of henselian rings) Let $(R, m, k)$ be a henselian ring.
   
   (a) Given a finite separable extension $i : k \to k'$, show that there is a unique henselian ring $(R', m', k')$ finite étale over $(R, m, k)$ such that $R \to R'$ reduces to $i$ modulo $m$.
   
   (b) Show that the association $k' \mapsto R'$ (from the previous exercise) extends to a sheaf on $\text{Spec}(k)_{\text{ét}}$.
   
   (c) Given a pointed scheme $(X, x)$ with $i : \text{Spec}(k(x)) \to X$ denoting the residue field, show that $\Gamma(\text{Spec}(k(x))_{\text{ét}}, i^{-1}\mathcal{O}_{X,x}) \simeq \mathcal{O}^h_{X,x}$.

4. Let $(R, m, k)$ be a local normal domain with fraction field $K$. Assume that $K$ is separably closed. Please do not assume $R$ is noetherian for this exercise.
   
   (a) Let $A \to B$ be a faithfully flat ring map of domains. If there is a factorisation $A \to B \to \text{Frac}(A)$ with $B \to \text{Frac}(A)$ injective, then show that $A = B$.
   
   (b) Show that any $R \to S$ étale can be written as $S = \prod_{i=1}^k S_i$ with each $R \to S_i$ an open immersion, and at least one $R \to S_i$ an isomorphism.
   
   (c) Show that $R$ is strictly henselian.

5. (The Nisnevich topology) Let $X$ be a noetherian scheme, and fix $n \in \mathbb{Z}$. Define the small Nisnevich site $X_{\text{Nis}}$ as follows: objects are étale maps $U \to X$, and a family $\{U_i \to U\}$ is a Nisnevich covering if it is an étale covering with the additional property that any field valued point $\text{Spec}(k) \to U$ lifts to some $U_i$. We use the subscript $\text{Nis}$ to denote cohomology in the Nisnevich topology.
(a) Show that descent and cohomology for quasi-coherent sheaves work as expected, i.e., quasi-coherent \( \mathcal{O}_X \)-modules define Nisnevich sheaves, and the Nisnevich cohomology of the resulting sheaves agrees with Zariski cohomology.

(b) Given a point \( x \in X \), let \( i : \text{Spec}(\kappa(x)) \to X \) denote the corresponding map. Show that \( \Gamma(\text{Spec}(k)_{\text{Nis}}, i^{-1}\mathcal{O}_{X_{\text{Nis}}}) \simeq \mathcal{O}_{X,x}^h \).

(c) Show that exactness of a sequence of Nisnevich sheaves can be detected by pullbacks along all maps of the form \( i : \text{Spec}(\kappa(x)) \to X \), where \( x \in X \) is a point.

(d) Let \( X \) be a normal connected scheme. Show that \( H^1_{\text{Nis}}(X, \mathbb{Z}/n) = 0 \). This shows that Nisnevich cohomology does not agree with \( \acute{e}tale \) cohomology in general.

(e) Let \( X \) be a normal connected scheme. Show that there is a finite morphism \( X \to_{\text{Nis}} Y \). Moreover, if \( f \) is finite, show that \( f_* \) is exact.

(f) Let \( X \subset \mathbb{P}^2 \) be an irreducible nodal cubic. Show that \( H^1_{\text{Nis}}(X, \mathbb{Z}/2) \simeq \mathbb{Z}/2 \) while \( H^1_{\text{Zar}}(X, \mathbb{Z}/2) \simeq 0 \). This shows that Nisnevich cohomology does not agree with Zariski cohomology in general.

6. (Finite morphisms, following Morel-Voevodsky) We will see that finite morphisms are acyclic for the Nisnevich topology (like \( \acute{e}tale \)), but not for the Zariski topology.

(a) Let \( f : X \to Y \) be a morphism of schemes. Show that there is a morphism of sites \( f : X_{\text{Nis}} \to Y_{\text{Nis}} \). Moreover, if \( f \) is finite, show that \( f_* \) is exact.

(b) Let \( f : X \to Y \) be a morphism of schemes, with \( Y \) the spectrum of a local ring. Show that the functor \( f : X_{\text{Zar}} \to Y_{\text{Zar}} \) induces an exact functor \( f_* \) if and only if \( H^i(X_{\text{Zar}}, F) = 0 \) for all \( F \in \text{Ab}(X_{\text{Zar}}) \) and \( i > 0 \).

Let \( X \) be the spectrum of the semilocal ring of \( \mathbb{A}^2 \) at two fixed points \( x_0 \) and \( x_1 \). Choose two irreducible curves \( C_1, C_2 \subset \mathbb{A}^2 \) such that \( C_1 \cap C_2 = \{ x_0, x_1 \} \). Let \( U \xrightarrow{j_1} V \xrightarrow{j_2} X \) be the sequence of open immersions defined by \( V = X - \{ x_0, x_1 \} \), and \( U = X - (C_1 \cup C_2) \); let \( j = j_2 \circ j_1 \).

(c) Using the Mayer-Vietoris sequence for the open cover \( V = (V - (C_1 \cap V)) \cup (V - (C_2 \cap V)) \), show that \( H(\text{V}_{\text{Zar}}, j_1, \mathbb{Z}) \) is non-zero.

(d) Using the Mayer-Vietoris sequence for the open cover \( X = (X - \{ x_0 \}) \cup (X - \{ x_1 \}) \), show that \( H^2(\text{X}_{\text{Zar}}, j, \mathbb{Z}) \) is non-zero.

(e) Show that there is a finite morphism \( X \to \text{Spec}(\mathcal{R}) \) for a local ring \( \mathcal{R} \). Conclude that finite morphisms need not be acyclic for the Zariski topology.

7. (Transfers) The goal of this exercise is to discuss the existence of norm maps in \( \acute{e}tale \) cohomology with coefficients in \( \mathbb{G}_m \).

(a) Let \( f : \mathcal{R} \to S \) be a finite locally free map of algebras. Given \( s \in S \), show that multiplication by \( s \) action of \( s \) on \( S \) gives a well-defined characteristic polynomial \( \phi_s \in \mathcal{R}[t] \) of degree \( \deg(f) \). In particular, show that there is a well-defined element \( \text{Nm}(s) \in \mathcal{R} \) such that \( \text{Nm}(f(r)) = r^{\deg(f)} \).

(b) Let \( f : Y \to X \) be a finite locally free map of schemes with \( X \) connected. Prove that there is a natural “norm” map \( \text{Nm}_f : f_*\mathbb{G}_m \to \mathbb{G}_m \) in \( \text{Ab}(\text{Sch}/X, \text{fppf}) \) such that the composite \( \mathbb{G}_m \xrightarrow{f_*} f_*f^*\mathbb{G}_m \xrightarrow{\text{Nm}_f} \mathbb{G}_m \) is \( x \mapsto x^{\deg(f)} \).
(c) Let \( f : Y \to X \) be a finite locally free map of schemes with \( X \) connected. Show that there is a natural map \( H^i(Nm_f) : H^i_{\text{fppf}}(Y, \mathbf{G}_m) \to H^i_{\text{fppf}}(X, \mathbf{G}_m) \) such that the composite

\[
H^i_{\text{fppf}}(X, \mathbf{G}_m) \xrightarrow{f^*} H^i_{\text{fppf}}(Y, \mathbf{G}_m) \xrightarrow{H^i(Nm_f)} H^i_{\text{fppf}}(X, \mathbf{G}_m)
\]

is multiplication by \( \deg(f) \), and similarly for étale cohomology.

(d) Show that for \( X = \text{Spec}(R) \) with \( R \) henselian local, the groups \( H^i_{\text{ét}}(k, \mathbf{G}_m) \) are torsion for \( i > 0 \). Can you give an example of a henselian ring \( R \) for which the torsion orders of \( H^2_{\text{ét}}(\text{Spec}(R), \mathbf{G}_m) \) can be arbitrarily large?

8. (More transfers) The goal of this exercise is to show norm maps exist for constant coefficients as well. As a corollary, we will see that étale cohomology with \( \mathbb{Q} \)-coefficients is typically uninteresting. Fix a scheme \( X \). Let \( A \) abusively denote the constant sheaf on \( X_{\text{ét}} \) associated to an abelian group \( A \).

(a) Let \( f : Y \to X \) be a finite étale map. Show that there is a norm map \( Nm_f : f_* f^* A \to A \) in \( \text{Shv}(X_{\text{ét}}) \) such that the composite \( A \xrightarrow{f_*} f_* f^* A \xrightarrow{Nm_f} A \) is multiplication by \( \deg(f) \).

(b) Assume that \( X \) is a henselian local scheme. Show that \( H^i_{\text{ét}}(X, A) \) is torsion for \( i > 0 \).

(c) Assume that \( X \) is a henselian local scheme. Show that \( H^i_{\text{ét}}(X, \mathbb{Q}) = 0 \) for \( i > 0 \). Deduce that \( H^1_{\text{ét}}(X, \mathbb{Z}) = 0 \) for \( X \) henselian.

(d) By contemplating singular curves, give an example of a local ring \( R \) such that \( H^1_{\text{ét}}(\text{Spec}(R), \mathbb{Z}) \) is non-zero.