Notation 0.1. A family \( \{ f_i : X_i \to X \} \) of maps is called an fpff cover if each \( f_i \) is locally finitely presented and flat, and \( \sqcup_i X_i \to X \) is surjective. An fpff cover is a Zariski cover if each \( X_i \to X \) is an open immersion.

1. The goal of this exercise is to give a criterion for testing when a presheaf is an fpff sheaf; the formulation below is meant to formalise the intuition that all maps are affine Zariski locally on the source and target. Fix a base scheme \( S \), and let \( F \in PShv(\text{Sch}_S) \). Show that \( F \) is an fpff sheaf if and only if the following two conditions are satisfied:

   (a) For \( T \in \text{Sch}_S \), and any Zariski cover \( \{ T_i \hookrightarrow T \} \), the sequence
   \[
   F(T) \to \prod_i F(T_i) \to \prod_{i \neq j} F(T_i \cap T_j).
   \]
   is exact.

   (b) For any fpff map \( \text{Spec}(B) \to \text{Spec}(A) \) of affine \( S \)-schemes, the sequence
   \[
   F(A) \to F(B) \to F(B \otimes_A B)
   \]
   is exact.

   Use this criterion to show that \( h_X \in PShv(\text{Sch}_S) \) is an fpff sheaf for any \( X \in \text{Sch}_S \).

2. Fix a base scheme \( S \). Let \( \mathcal{A}ff_S \subset \text{Sch}_S \) be the full subcategory spanned by maps \( X \to S \) with \( X \) affine; equip \( \mathcal{A}ff_S \) with the fpff topology, i.e., a family of maps in \( \mathcal{A}ff_S \) is an fpff cover if it is so after applying \( u \). Show that there is a morphism of sites \( \text{Sch}_S, \text{fpff} \to \mathcal{A}ff_S, \text{fpff} \) that induces an equivalence \( \text{Shv}(\text{Sch}_S, \text{fpff}) \simeq \text{Shv}(\mathcal{A}ff_S, \text{fpff}) \).

3. (Descent for quasi-affine maps) Let \( f : T \to S \) be a faithfully flat and finitely presented map of schemes. Let \( F \in \text{Shv}(\text{Sch}_S, \text{fpff}) \) be such that \( f^* F \) is representable by quasi-affine morphism \( X \to T \). Show that \( f \) is representable by a quasi-affine morphism.

4. The goal of this exercise is to show that cohomology groups can be computed in either the big or the small topologies without affecting the answer. Fix a base scheme \( X \). Let \( \mathcal{C} = \text{Sch}_{X, \text{ét}} \) and \( \mathcal{D} = X_{\text{ét}} \) denote the big and small sites of \( X \) respectively.

   (a) Show that there is a morphism of sites \( f : \mathcal{C} \to \mathcal{D} \).

   (b) Show that \( f_* : \text{Ab}(\mathcal{C}) \to \text{Ab}(\mathcal{D}) \) is exact.

   (c) Show that \( f_* \) preserves injectives and resolutions, and hence injective resolutions.

   (d) Conclude that for any \( F \in \text{Sch}_{X, \text{ét}} \), there is a natural isomorphism \( H^*(X_{\text{ét}}, f_* F) \simeq H^*(X, F) \), where we view the right (resp. left) hand side as cohomology in \( \mathcal{C} \) (resp. \( \mathcal{D} \)).
5. (Cech description of \( H^1 \)) Let \( \mathcal{C} \) be a Grothendieck topology, and let \( G \in \text{Ab}(\mathcal{C}) \).
   
   (a) Show that for any \( U \in \mathcal{C} \), and \( \alpha \in H^k(U,G) \) with \( k > 0 \), there exists a cover \( \{ f_i : U_i \to U \} \) such that \( f_i^* \alpha = 0 \). (Use injective resolutions)
   
   (b) Show that there is a natural isomorphism \( H^1(U,G) \cong H^1(U,G) \) for any \( U \in \mathcal{C} \). (Use the spectral sequence)

6. Let \( \mathcal{C} \) be a Grothendieck topology, and let \( G \in \text{Shv}(\mathcal{C}) \) be a sheaf of (not necessarily abelian) groups. Show that the formula defining the first Cech cohomology group \( H^1(U,G) \) still makes sense, but gives a pointed set, rather than a group. When \( \mathcal{C} \) is the category of \( H \)-sets for some group \( H \), this is nicely discussed in the Appendix to Chapter VII in Serre’s *Local fields* book.

7. (Torsors) Let \( \mathcal{C} \) be a Grothendieck topology with a final object \( * \), and let \( G \in \text{Shv}(\mathcal{C}) \) be a sheaf of groups. A \( G \)-torsor is a sheaf \( F \in \text{Shv}(\mathcal{C}) \) together with a simply transitive action of \( G \) such that \( F(U_i) \neq \emptyset \) for some cover \( \{ U_i \to * \} \); here simply transitive means that for any \( T \in \mathcal{C} \) and \( g \in F(T) \), the orbit map \( G(T) \to F(T) \) is an isomorphism.
   
   (a) Let \( H \) be a sheaf of sets with a \( G \)-action. Show that \( H \) is a \( G \)-torsor if and only if the map \( G \times H \to H \times H \) defined by \( (g,h) \mapsto (gh,h) \) is an isomorphism, and \( H \) has sections over some cover of \( * \).
   
   (b) Given a morphism \( f : \mathcal{D} \to \mathcal{C} \) of sites (with final objects), show that \( G \)-torsors pullback to \( f^* G \)-torsors in the obvious sense.
   
   (c) Show that there is a bijection between \( H^1(\mathcal{C},G) \) and isomorphism classes of \( G \)-torsors, and that this bijection respects pullbacks along morphisms of sites.

8. Let \( \mathcal{C} = \text{Sch}_{/k,\text{fppf}} \), and let \( G \to \text{Spec}(k) \) be a group scheme, i.e., a \( k \)-scheme such that \( \text{Hom}_k(\mathcal{O},G) \) is valued in groups. Examples include \( G = X_n, G_m, \mu_n, GL_n \) and abelian varieties.
   
   (a) Show that if \( G \) is affine, then any \( G \)-torsor is representable by an affine scheme, i.e., if a sheaf \( F \) is a \( G \)-torsor, then \( F = \text{Hom}_k(\mathcal{O},X) \) for some affine \( k \)-scheme \( X \) with a \( G \)-action.
   
   (b) Show the converse to the above statement, i.e., show that if a \( G \)-torsor \( F \) is representable by an affine \( k \)-scheme \( X \), then \( G \) is itself affine.
   
   (c) Compute \( H^1_{\text{fppf}}(\text{Spec}(\mathcal{O}_q),\mathbb{Z}/n) \) and \( H^1_{\text{fppf}}(\mathcal{O}_q^1,\mathbb{Z}/n) \) for any \( n \).

9. (Galois descent) Let \( T \) be a scheme with a \( G \)-action. Assume that there is a finite group \( G \) acting on \( T \). Recall that a \( G \)-equivariant sheaf on \( T \) is the data of a sheaf \( E \in \text{Shv}(T_{\text{et}}) \) together with isomorphisms \( \sigma_g : g^* E \cong E \) for each \( g \in G \) such that \( \sigma_{gh} = \sigma_h \circ h^*(\sigma_g) \). The category of all such sheaves is denoted \( \text{Shv}(T_{\text{et}})^G \). Let \( f : T \to S \) be a \( G \)-invariant morphism.
   
   (a) Show that pullback defines a functor \( \overline{f}^* : \text{Shv}(S_{\text{et}}) \to \text{Shv}(T_{\text{et}})^G \).
   
   (b) Assume \( T \) is an étale \( G \)-torsor over \( S \) via \( f \). Show that \( \overline{f}^* \) is an equivalence.
   
   (c) What does the previous item say for \( f \) being the map \( \text{Spec}(\mathcal{O}) \to \text{Spec}(\mathbb{R}) \) with \( G = \mathbb{Z}/2 \)?

10. (Special case of Grothendieck’s theorem) Let \( G \to S \) be a smooth affine group scheme, and let \( F \in \text{Shv}(\text{Sch}_{/S,\text{fppf}}) \) be a \( G \)-torsor. Show that there exists an étale cover \( \{ S_i \to S \} \) such that \( F(S_i) \neq \emptyset \). (The point here is that we only assume existence of sections over an fppf cover, and deduce sections over an étale cover)
11. (Twisted forms) Let $\mathcal{C}$ be a Grothendieck topology with a final object $\ast$, and let $F \in \text{Shv}(\mathcal{C})$. A twisted form of $F$ is a sheaf $G \in \text{Shv}(\mathcal{C})$ which is isomorphic to $F$ over some cover of $\ast$.

(a) Construct a bijection between $\check{H}^1(\ast, \text{Aut}(F))$ and isomorphism classes of twisted forms of $F$.

(b) The identity map $\check{H}^1(\ast, \text{Aut}(F)) \to \check{H}^1(\ast, \text{Aut}(F))$ can be viewed as a procedure for associating $\text{Aut}(F)$-torsors to twisted forms of $F$. What is this procedure explicitly?

(c) Assume that $F \in \text{Shv}(\mathcal{C})$ is a sheaf of groups. Then the left action $F$ on itself defines a group homomorphism $F \to \text{Aut}(F)$, and so a map $H^1(\ast, F) \to H^1(\ast, \text{Aut}(F))$. What is this map in terms of torsors and twisted forms?

(d) Now assume $F \in \text{Mod}(R)$ for some sheaf of rings $R \in \text{Shv}(\mathcal{C})$. Construct a bijection between $\check{H}^1(\ast, \text{Aut}_R(F))$ and twisted forms of $F$ as an $R$-module, i.e., $R$-modules $M$ which are locally isomorphic to $F$ as $R$-modules.

12. (Non-abelian Hilbert 90) Let $X$ be a scheme, and let $\mathcal{C} = \text{Sch}_{/X, \text{fppf}}$.

(a) Show that the functor $U \mapsto \text{GL}_n(\Gamma(U, \mathcal{O}_U))$ is representable by a group scheme. In particular, it defines a sheaf of groups on $\mathcal{C}$. We call the representing scheme and the sheaf $\text{GL}_n$.

(b) Show that there is a natural bijection between $\check{H}^1_{\text{fppf}}(U, \text{GL}_n)$, $\check{H}^1_{\text{ét}}(U, \text{GL}_n)$, $\check{H}^1_{\text{Zar}}(U, \text{GL}_n)$, and the isomorphism classes of rank $n$ vector bundles on $U$ (in the classical sense); here the subscript is telling us the site to compute Čech cohomology in.

(c) What is the $\text{GL}_n$-torsor associated to a rank $n$ vector bundle (in any of the above topologies)?

13. Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites, and let $F \in \text{Ab}(\mathcal{C})$. Show that $R^i f_* F$ is the sheafification of the presheaf $U \mapsto H^i(uf(U), F)$ where $uf : \mathcal{D} \to \mathcal{C}$ is the continuous functor underlying $f$.

14. Fix a field $k$. Consider the presheaf $F \in \text{PShv}(\text{Sch}_{/k})$ defined by $F(U) = \Gamma(U, \Omega^1_{U/k})$.

(a) Show that $F$ is an étale sheaf.

(b) Give an example to show that $F$ is not an fppf sheaf.

(c) (*) What happens in the smooth topology?

15. The goal of this exercise is to discuss certain strange phenomena that occur on small sites. Let $k$ be a field of characteristic $p$. Let $X_0$ be a reduced $k$-scheme, and let $X = X_0 \times_k \text{Spec}(k[\epsilon])$. Let $i : X_0 \to X$ be the closed immersion defined by $\epsilon \mapsto 0$.

(a) Show that $i^*$ induces an equivalence $\text{Shv}(X_{\text{ét}}) \simeq \text{Shv}(X_{0, \text{ét}})$. (Use the equivalence between $X_{0, \text{ét}}$ and $X_{\text{ét}}$ from a previous problem set)

(b) Show that $H^i(X_{0, \text{ét}}, \mu_p) = 0$ for all values of $i$.

(c) Show that $i^*(\mu_p) = G_n$. (This is not a contradiction to a previous exercise because $\mu_p$ is not representable on the small site $X_{\text{ét}}$)

(d) Give an example to show that $H^i(X_{\text{ét}}, \mu_p)$ is not always 0.

(e) Check for yourself that the preceding phenomenon does not occur in characteristic 0: the group scheme $\mu_p$ is representable on the small site in that case.