Problem set 2

1. Give an example of a ring map $A \rightarrow B$ which satisfies the infinitesimal lifting criterion for being étale, but is not flat itself.
   
   Hint: Consider the ring $k[\mathbb{Q}_{\geq 0}]$ for some characteristic $p$ field $k$ and build a map.

2. Give an example of a flat ring map $A \rightarrow B$ such that $\Omega^1_{B/A} = 0$, but $A \rightarrow B$ does not satisfy the infinitesimal lifting criterion for being étale. Note that such a map cannot be finitely presented.
   
   Hint: try playing with quotients of perfect rings in characteristic $p$.

3. Let $f : X \rightarrow S$ be a locally finitely presented unramified morphism. Show that any section of $f$ is an open immersion. If $f$ is additionally assumed to be separated, then show that a section has to be an isomorphism onto a connected component.
   
   Hint: Consider the map $X \rightarrow X \times_S X$ defined by $(s \circ f, \text{id})$ where $s : S \rightarrow X$ is a section of $f$, and use that $\delta : X \rightarrow X \times_S X$ is an open immersion if $f$ is unramified (and also closed if $f$ is separated).

4. A morphism $f : X \rightarrow S$ is called a finite étale cover if $f$ is finite, surjective, and étale. Classify all finite étale covers of:
   
   (a) $\text{Spec}(\mathbb{R})$ where $\mathbb{R}$ is the field of real numbers.
   (b) $\mathbb{P}^1_\mathbb{C}$ and $\mathbb{P}^1_{\mathbb{F}_p}$
   (c) $\mathbb{A}^1_\mathbb{C} = \text{Spec}(\mathbb{C}[t])$.
   (d) $\mathbb{G}_{m, \mathbb{C}} = \text{Spec}(\mathbb{C}[t, t^{-1}])$.
   (e) $\text{Spec}(O_{C,x})$ where $C$ is a smooth projective curve over $\mathbb{C}$, and $x \in C$ is a closed point. Answer in terms of the projective geometry of $C$.
   (f) An artinian local $\mathbb{C}$-algebra.
   (g) A complete local $\mathbb{C}$-algebra.
   (h) An elliptic curve $E$ over $\mathbb{C}$.
   (i) The nodal cubic in $\mathbb{P}^2_\mathbb{C}$.
   (j) The cuspidal cubic in $\mathbb{P}^2_\mathbb{C}$. (The next exercise may be useful here.)
   (k) $\text{Spec}(\mathbb{Z})$.
   (l) $\text{Spec}(\mathbb{Z}_{(p)})$, the local scheme of $\text{Spec}(\mathbb{Z})$ at a prime $p$. Answer in terms of number fields.
   (m) $\text{Spec}(\mathbb{Z}_p)$, the completion of $\text{Spec}(\mathbb{Z})$ at a prime $p$.
   (n) (*) $\mathbb{A}^2_\mathbb{C} - \{0\}$
5. Let \( S \) be a noetherian, integral, normal, excellent scheme; you may take \( S \) to be a variety if you wish. Let \( U \subset S \) be a dense open subset. Show that the restriction functor \( X \mapsto X_U := X \times_S U \) is fully faithful on the category of finite étale covers of \( S \). Give an example to show that the normality condition cannot be dropped.

Hint: For the first part, “normalise.” For the example, see list above.

6. Let \( k \) be a field, and let \( X \) be a \( k \)-scheme of finite type. Show that \( X \) is smooth over \( k \) of dimension \( d \) if and only if for every affine \( U \subset X \), the ring \( \mathcal{O}(U) \) is geometrically regular of dimension \( d \), i.e., for any field extension \( L/k \), the base change \( \mathcal{O}(U) \otimes_k L \) is a regular ring of dimension \( d \).

For the forward direction, use the fact that, locally around \( x \), \( X \) is étale over affine space. For the backwards direction, reduce to the case that \( k \) is algebraically closed.

7. (Local structure for étale morphisms) Let \( f : X \to S \) be a locally finitely presented morphism which is étale at \( x \in X \). Then show that, Zariski locally on \( S \) around \( f(x) \), there exists a presentation \( \mathcal{O}_{X,x} \cong (\mathcal{O}_S[t]/(f))^p \), with \( f \) monic, and \( p \in \text{Spec}(\mathcal{O}_S[t]/(f)) \) some prime where \( f' \) is invertible.

Solution sketch: using Zariski’s main theorem, realise \( f \) as \( X \subset X' \to S \) with \( X' \to S \) finite and \( X \subset X' \) an open immersion. May work locally on \( S \), so we may assume \( S \) is local. Now \( \kappa(x)/\kappa(s) \) has a single generator \( \overline{g} \) by the primitive element theorem. Lift this generator to an element \( g \in \mathcal{O}_{X,x} \) with the property that its image in all other residue fields lying above \( s \) is \( 0 \); we can do this by the Chinese remainder theorem. This generator defines a map \( X \subset X' \to \mathbb{A}^1_S \). Choose some monic polynomial \( f \in \mathcal{O}_S[t] \) that \( g \in X \) satisfies. Then the map factors as \( X \subset X' \to \text{Spec}(\mathcal{O}_S[t]/(f)) \). Now observe that by the choice of \( g \), this last map has the property that \( i^{-1}(i(x)) = x \) (because the element \( t \) is invertible in \( \kappa(i(x)) \), and hence also for any point in the preimage). Hence, by Nakayama applied to \( \mathcal{O}_S[t]/(f) \), conclude that the map \( \mathcal{O}_S[t]/(f) \to \mathcal{O}_{X,x} \) is surjective. Since \( g \) a primitive element at \( x \), there is a factorisation \( \mathcal{O}_S[t]/(f) \to (\mathcal{O}_S[t]/(f))_{f'} \to \mathcal{O}_{X,x} \). This map is étale since both the source and target are so. Hence, the composite \( \text{Spec}(\mathcal{O}_S[t]/(f))_{f'} \to \text{Spec}(\mathcal{O}_S[t]/(f)) \subset \text{Spec}(\mathcal{O}_S[t]/(f(t))) \) is an immersion at \( x \in X \) that is also étale. Show that this forces the map to be an open immersion.

8. Let \( f : (A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B) \) be a local homomorphism of noetherian local rings. Assume that \( f \) is essentially finitely presented and étale. Then show

(a) \( A \) is regular if and only if \( B \) is so.
(b) \( A \) is Cohen-Macaulay if and only if \( B \) is so.
(c) \( A \) is Gorenstein if and only if \( B \) is so.
(d) \( A \) is an integral domain if \( B \) is so; the converse fails (example?).

Hint: use the characterisation of étale ring maps as flat and unramified ones, and homological criteria.

9. Let \( S \) be a scheme, and \( S_0 \subset S \) be a closed subscheme defined by a quasi-coherent ideal of square zero. The goal of this exercise is to show that the functor \( X \mapsto X_0 := X \times_S S_0 \) induces an equivalence between categories of étale schemes over \( S \) and \( S_0 \).

(a) Show using the infinitesimal lifting criterion that if \( X \) and \( Y \) are \( S \)-schemes with \( X \) étale, then \( \text{Hom}_S(Y, X) \cong \text{Hom}_{S_0}(Y_0, X_0) \). Note that \( Y \) is not assumed to be affine.

(b) Show that if \( X_0 \) is a standard étale \( S_0 \)-scheme (i.e., one of the form \( \text{Spec}((\mathcal{O}_{S_0}[t]/(f)))_{f'} \) with \( f \) monic), then \( X_0 \) comes from an étale \( S \)-scheme \( X \). Moreover, show that \( X \) is determined uniquely up to unique isomorphism once \( X_0 \) is specified.
(c) Using the uniqueness in the previous step, show by glueing that all étale $S_0$-schemes arise from those over $S$ by base change.

10. The goal of this exercise is to introduce some ideas of deformation theory, putting the preceding exercise in some perspective. Let $S_0 \subset S$ be a closed immersion of affine schemes defined a quasi-coherent ideal $\mathcal{I}$ of square zero on a scheme $S$. Let $f_0 : X_0 \to S_0$ be a smooth finitely presented morphism. A deformation of $f_0$ to $S$ is defined to be a pair $(f,i)$ where $f : X \to S$ is a flat morphism, and $i : X_0 \to X \times_S S_0$ is an $S_0$-isomorphism; we sometimes abusively refer to $f$ as the deformation. Let $\text{Def}(f_0, S)$ denote the category of all deformations of $f_0$ to $S$, and let $\pi_0(\text{Def}(f_0, S))$ denote the set of isomorphism classes.

(a) Show that if $f_0$ arises via base change from a flat morphism $f : X \to S$, then $f$ has to be smooth.

(b) Show that $\text{Def}(f_0, S)$ is a groupoid, i.e., any morphism in $\text{Def}(f_0, S)$ is an automorphism.

(c) If $f_0$ admits a deformation $f : X \to S$, then show that the kernel of $\text{Aut}_S(X) \to \text{Aut}_{S_0}(X_0)$ is identified naturally with $\text{Hom}_{X_0}(\Omega^1_{X_0/S_0}, f_0^*\mathcal{I})$. Note that this is simply the automorphism group in the category $\text{Def}(f_0, S)$ at the point defined by $f$, and turns out to be independent of $f$.

(d) If $f_0$ is affine, then show that $\text{Def}(f_0, S)$ is non-empty and connected, i.e., there exists a deformation $f : X \to S$, and any two deformations are isomorphic.

Hint: To show non-emptiness, embed in affine space, and lift a set of well-chosen equations. To show uniqueness (up to non-unique isomorphism), use the infinitesimal lifting property to lift the identity map to an isomorphism between any two deformations.

(e) Show that $\pi_0(\text{Def}(f_0, S))$ is a torsor for $\text{Ext}^1_{X_0}(\Omega^1_{X_0/S_0}, f_0^*\mathcal{I})$, i.e., the latter group acts simply transitively on the former set, provided this set is non-empty.

Hint: Given two deformations $f_1$ and $f_2$, pick an affine chart for $X_0$, write down isomorphisms between $f_1$ and $f_2$ over those charts using the previous claim, and see what you get.

(f) (*) Give an example to show that $\text{Def}(f_0, S)$ could be empty.

(g) Show that there exists a canonical element $\text{ob} \in \text{Ext}^2_{X_0}(\Omega^1_{X_0/S_0}, f_0^*\mathcal{I})$ which is zero if and only if $\text{Def}(f_0, S)$ is non-empty.

Hint: Pick an affine chart for $X_0$, deformations of pieces of the chart to $S$, and isomorphisms over the overlaps. What is the obstruction to glueing?

(h) Observe what happens when $f_0$ is étale.

11. The goal of this exercise is to introduce some ideas that become useful when thinking about the cotangent complex. Let $A \to B$ be a ring homomorphism. Let $\mathcal{C}$ be the category of $A$-algebras equipped with a map to $B$. For a $B$-module $M$, let $B \oplus M$ denote the square-zero extension of $B$ by $M$, i.e., we define multiplication by

$$(b_1,m_1) \cdot (b_2,m_2) = (b_1b_2, b_1m_2 + b_2m_1).$$

(a) Show that the association $M \mapsto B \oplus M$ defines a functor $\mathcal{F} : \text{Mod}(B) \to \mathcal{C}$. Observe that the two maps $0 \to M$ and $M \to 0$ define maps $B \simeq \mathcal{F}(0) \to \mathcal{F}(M)$ and $\mathcal{F}(M) \to \mathcal{F}(0) \simeq B$ for any $B$-module $M$.

(b) Show that for any $R \in \mathcal{C}$, there is a natural isomorphism $\text{Hom}_\mathcal{C}(R, \mathcal{F}(M)) \simeq \text{Der}_A(R, M)$. Conclude that if $A \to B$ satisfies the infinitesimal lifting criterion for smoothness, then $\Omega^1_{B/A}$ is a projective $B$-module. Note the absence of finiteness conditions.
(c) (*) Observe that $\mathcal{F}(M)$ is an abelian group object of $\mathcal{C}$ (i.e., that $\text{Hom}_\mathcal{C}(-, \mathcal{F}(M))$ is naturally valued in abelian groups) thanks to the formula above in terms of derivations. Show that every abelian group object in $\mathcal{C}$ is (uniquely, up to unique isomorphism) of this form, i.e., that $\mathcal{F}$ factors as an equivalence $\text{Mod}(B) \simeq \text{Ab}(\mathcal{C})$ followed by the forgetful functor $\text{Ab}(\mathcal{C}) \to \mathcal{C}$.

(d) Reinterpret the preceding results to show that the forgetful functor $\text{Ab}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint “abelianisation” functor $(-)^{\text{Ab}} : \mathcal{C} \to \text{Ab}(\mathcal{C})$ defined by $R \mapsto R^{\text{Ab}} := \mathcal{F}(\Omega_{R/A}^1 \otimes_R B)$. Note that you could also have predicted the existence of $(-)^{\text{Ab}}$, and hence that of theory of Kahler differentials, by simply observing that $\mathcal{F}$ preserves limits, and applying the adjoint functor theorem.