TOPICS IN ALGEBRAIC GEOMETRY: PERVERSE SHEAVES (MATH 731)

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Goal. The goal of this class is to introduce and study perverse sheaves (and thus intersection cohomology) on algebraic varieties, and discuss related results and applications.

Background. In the 70s, Goresky and Macpherson developed a new homology theory for a fairly large class of topological spaces X (including all complex algebraic varieties) called *intersection homology* $\mathrm{IH}^*(X)$. Their theory has many advantages over ordinary homology on singular spaces; for example, $\mathrm{IH}^*(X)$ always satisfies Poincare duality, and thus supports a workable intersection theory. At a Halloween party, Deligne reinterpreted $\mathrm{IH}^*(X)$ as the hypercohomology of the *intersection cohomology complex* $\mathbf{IC}_X$; the latter is an explicit object of the derived category $D(X, \mathbf{Q})$ of sheaves on X, and can be built from a suitable stratification of X using the pushforward and truncation operations. His description reaped two instant rewards: (a) it made $\mathrm{IH}^*(X)$ accessible to sheaf-theoretic and derived category techniques; for example, Poincare duality for $\mathrm{IH}^*(X)$ results from an autoduality of $\mathbf{IC}_X$, and can thus be proven locally, and (b) it made intersection cohomology available in positive characteristic algebraic geometry. Soon thereafter, the complex $\mathbf{IC}_X$ was found to lie in (and, in fact, generate, in a suitable sense) a remarkably nice *abelian* category living inside $D(X, \mathbf{Q})$; this is now called the category $\mathrm{Perv}(X)$ of *perverse sheaves*.

A few years later, in the early 80s, Beilinson-Bernstein-Deligne(-Gabber) developed the theory of perverse sheaves in the purely algebraic context of the étale topology, and proved the decisive *decomposition theorem*. This result, which builds on Deligne's fundamental "Weil II" paper, is a vast generalization of an older result of Deligne — the Leray spectral sequence degenerates for a smooth and proper morphism — to arbitrary proper morphisms. This theorem, whose formulation requires perverse sheaves, has proven remarkably effective understanding singular varieties and morphisms, and has applications far beyond algebraic geometry; Kleiman[1] calls these results "some of the deepest theorems in algebraic geometry, if not all of mathematics."

Plan. We will spend roughly half the semester on the basics necessary to get the theory of perverse sheaves off-the-ground. This includes:

- Derived categories and t-structures.
- Recollection on the formalism of the 6 functors.
- Construction and basic properties of perverse sheaves.
- The yoga of weights and the decomposition theorem (statements).

In the remaining time, we will focus on applications. A sample superset of potential topics is:

- The derived category of perverse sheaves (following Beilinson) and the geometric description of the Leray and perverse Leray filtrations (following Arapura and De Cataldo-Migliorini).
- The Radon transform for perverse sheaves on projective space, and its applications to understanding the topology of families of hyperplane sections on a projective variety (following Brylinski and Beilinson).
- Some geometric ideas entering the proof of the decomposition theorem (following BBD).

Background. I will assume knowledge of algebraic geometry at the level of Hartshorne's book. Familiarity with constructible sheaf theory (either in the topological or étale contexts) will be extremely useful.

Office hours. TBD

Homework. There will be problem sets posted on the course website
`http://www-personal.umich.edu/~bhattb/mat731f15/`

References. The canonical reference is "Faisceaux pervers" by Beilinson-Bernstein-Deligne (aka BBD). Others references can be found on the course webpage, and will be given in class.

[1] See §7 of his paper *The development of intersection homology theory* on the arXiv.