Perverse sheaves: Problem set 5

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We follow the notation used in class.

1. Let $f : X \to Y$ be a smooth morphism with fibres of equidimension d. Show that $f^*\mathrm{IC}_Y[d] \simeq \mathrm{IC}_X$.

For the next problem, the following form of Deligne's generic base change result might be useful:

Theorem 0.1 (Deligne)**.** *Let S be a variety, and let $f : X \to Y$ be a map of S-varieties. Fix some $K \in D(X)$. Then there exists a dense open $U \subset S$ such that the formation of Rf_*K commutes with base changes $T \to U \subset S$.*

2. Let $\mathbf{P}$ be a projective space of dimension n with dual projective space $\mathbf{P}^\vee$. Let $X \subset \mathbf{P}$ be a projective variety of dimension d, and let $j : U \hookrightarrow X$ be an affine open subset with U smooth. Let $\mathcal{H} \subset X \times \mathbf{P}^\vee$ be the universal family of hyperplanes in X.
   (a) Show that the projection $\pi : \mathcal{H} \to X$ is a projective space bundle of relative dimension $d - 1$.
   (b) Let $\mathcal{H}_U = \mathcal{H} \times_X U$, and let $j_{\mathcal{H}_U} : \mathcal{H}_U \to \mathcal{H}$ be the induced affine open immersion. Show the following diagram commutes
$$\begin{array}{ccc} \mathrm{Perv}(U) & \longrightarrow & \mathrm{Perv}(\mathcal{H}_U) \\ \downarrow & & \downarrow \\ \mathrm{Perv}(X) & \longrightarrow & \mathrm{Perv}(H) \end{array}$$
   Here the horizontal functors are given by $\pi^*[d-1]$, while the vertical ones are either both !-pushforward or both $*$-pushforward along j and $j_{\mathcal{H}_U}$ respectively.
   (c) Show that $\pi^*\mathrm{IC}_X[d-1] \simeq \mathrm{IC}_{\mathcal{H}}$.
   (d) Show that there exists some dense open $V \subset \mathbf{P}^\vee$ such that for each $v \in V$ corresponding to a hyperplane $H_v \subset X$, we have $\mathrm{IC}_X|_H \simeq \mathrm{IC}_{H_v}[1]$.

For the next two problems, we adopt the following definition:

Definition 0.2. For an algebraic variety X, set $IH^i(X) = H^i(X, \mathrm{IC}_X)$, and $IH^i_c(X) = H^i_c(X, \mathrm{IC}_X)$.

3. For any variety X, show that $IH^i(X)$ is dual to $IH_c^{-i}(X)$ for all i.

4. Let X be an proper variety with an isolated singularity at $x \in X$. Set $U := X - \{x\} \overset{j}{\hookrightarrow} X$, and $Z := \{x\} \overset{i}{\hookrightarrow} X$. Let $d = \dim(X)$.
   (a) Show that there are natural maps $H_c^{i+d}(U) \to IH^i(X) \to H^{i+d}(U)$ such that the composite is the canonical "forget supports" map.
   (b) Show that $i^*\mathrm{IC}_X \in D^{\leq -1}(Z)$, and conclude that $H_c^{i+d}(U) \to IH^i(X)$ is surjective for $i = 0$, and bijective for $i > 0$.
   (c) Show that $i^!\mathrm{IC}_X \in D^{\geq 1}(Z)$, and conclude that $IH^i(X) \to H^{i+d}(U)$ is injective for $i = 0$, and bijective for $i < 0$.
   (d) Show that $IH^0(X)$ is the image of $H_c^d(U) \to H^d(U)$.
   (e) Show that $\mathrm{IC}_X \simeq \tau^{\leq -1}(j_*\mathbf{Z}/\ell[d])$. (Hint: check that i^* and $i^!$ behave as expected.)