Perverse sheaves: Problem set 4

Bhargav Bhatt

In the lectures, we have seen how to define "perverse" sheaves on a space equipped with a 1-step stratification (i.e., $X = U \sqcup Z$ with U open and Z closed) by glueing t-structures on the strata. The goal of this problem set is to inductively define perverse sheaves on more general stratified spaces, and give Deligne's formula for intermediate extensions, following §2.1 in BBD.

Let X be a topological space equipped with a finite decomposition $X = \sqcup_s X_s$ into locally closed subsets X_s indexed by a finite set S; each X_s is called a *stratum* of X. Assume that the closure of each stratum X_s is a union of strata. Fix a function $p : S \to \mathbf{Z}$; this is called the *perversity* function. Let $i_s : X_s \to X$ denote the inclusion of X_s.

1. Show that the pushforward[1] functor $i_{s,*}$ admits a left-adjoint i_s^*.
2. Show that there is an "extension by 0" functor $i_{s,!} : D(X_s) \to D(X)$ (specializing to $i_{s,*}$ is X_s is closed in X, and $i_{s,!}$ if X_s is open) that admits a right-adjoint $i_s^!$.

This allows us to make the following definition:

Definition 0.1. Let ${}^pD^{\leq 0}(X)$ be the full subcategory of $D(X)$ spanned by those K such that $i_s^*K \in D^{\leq p(s)}(X_s)$ for each s. Dually, let ${}^pD^{\geq 0}(X)$ be the full subcategory of $D(X)$ spanned by those K such that $i_s^!K \in D^{\geq p(s)}(X_s)$.

3. Using induction of $\#S$ and the glueing explained in class when $\#S = 2$, show that $({}^pD^{\leq 0}(X), {}^pD^{\geq 0}(X))$ gives a t-structure on $D(X)$.

Definition 0.2. Let $\mathrm{Perv}(X, S, p)$ be the heart of the t-structure defined above; this is the abelian category of *perverse sheaves* with respect to the given stratification and perversity functions[2]. If the data of the stratification and perversity function are clear, we simply write $\mathrm{Perv}(X)$ instead. This t-structure gives *perverse truncation* functors ${}^p\tau^{\leq i} : D(X) \to {}^pD^{\geq i}(X)$, ${}^p\tau^{\geq i} : D(X) \to {}^pD^{\geq i}(X)$, and *perverse homology* functors ${}^pH^i : D(X) \to \mathrm{Perv}(X)$.

For the rest of the problem set, let $j : U \to X$ be the inclusion of a union of strata, so j is locally closed. We have functors $j_!$, $j^!$, j_*, and j^* as before.

4. Show that $j_!$ and j^* are t-exact on the right, and deduce that $j^!$ and j_* are t-exact on the left (for the t-structure above).

By restricting the given data to U, we obtain a notion of perverse sheaves on U (and thus an abelian category $\mathrm{Perv}(U)$). By passing to perverse homology, as in class, we also have perverse analogs ${}^pj_!$, ${}^pj_*$, and ${}^pj^*$ of the standard functors between $\mathrm{Perv}(U)$ and $\mathrm{Perv}(X)$; for example, ${}^pj_!A := {}^pH^0(j_!A)$.

5. Check that $({}^pj_!, {}^pj^!)$ and $({}^pj^*, {}^pj_*)$ form adjoint pairs.
6. Using the canonical map $j_! \to j_*$, define a natural map ${}^pj_! \to {}^pj_*$ of functors $\mathrm{Perv}(U) \to \mathrm{Perv}(X)$.

With this notation, we can define:

[1] In this problem set, all functors are derived, unless we denote otherwise. Thus, we write f_* instead of Rf_*, etc. Also, if you prefer, you may assume that the relevant functors have finite cohomological dimension, and work exclusively in the bounded setting.

[2] Later, we will see that for algebraic varieties, there is an intrinsic notion of "perverse sheaves", independent of the choice of stratification, and with a canonical choice of perversity (the so-called *middle* perversity).

Definition 0.3. Let $j_{!*} : \mathrm{Perv}(U) \to \mathrm{Perv}(X)$ be the functor sending $A \in \mathrm{Perv}(U)$ to the image of ${}^p j_! A \to {}^p j_* A$.

7. Let $k : V \to U$ be the inclusion of a union of strata. Show that $(j \circ k)_{!*} \simeq j_{!*} \circ k_{!*}$.

8. Assume that U is open in X. For any $A \in \mathrm{Perv}(U)$, show that $j_{!*}A$ is the unique extension of A to $\mathrm{Perv}(X)$ such that for each stratum $X_s \in X - U$, we have $i_s^* A \in D^{\leq p(s)-1}(X)$, and $i_s^! A \in D^{\geq p(s)+1}(X)$.

Now assume that the perversity function p is semicontinuous in the following sense: $p(s) \geq p(t)$ if $X_s \in \overline{X_t}$. For each integer n, let $F_n \subset X$ be the union of those X_s for which $p(s) \geq n$; likewise, let $U_n \subset X$ be the union of those X_t for which $p(t) \leq n$.

9. Show that F_n is closed, and U_n is open.

We thus obtain an increasing union

$$\emptyset =: U_{-\infty} \subset ... \subset U_i \subset U_{i+1} \subset ... \subset U_\infty := X$$

of open subsets of X. Let $j_n : U_{n-1} \to U_n$ be the corresponding inclusion. Fix some integer k, some $A \in \mathrm{Perv}(U_k)$, some integer $a \geq k$ such that $p(s) \leq a$ for all s. Thus, $X = U_a$, and let $j : U_k \to U_a$ be the corresponding inclusion.

10. For any $A \in \mathrm{Perv}(U_k)$, show that

$$j_{!*}A = \left(\tau^{\leq a-1} j_{a,*}\right) \circ ... \circ \left(\tau^{\leq k+1} j_{k+2,*}\right) \circ \left(\tau^{\leq k} j_{k+1,*}\right)(A).$$