Perverse sheaves: Problem set 3

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For a noetherian ring R, write $D^b_f(R)$ for the full subcategory of $D(R)$ spanned by bounded complexes K with finitely generated homology.

1. Let $\mathcal{D}$ be a triangulated category equipped with t-structure. Show that $\mathrm{Ext}^i_{\mathcal{D}}(X, Y) = 0$ for $i < 0$ if X and Y are in the heart.

2. Let X be a topological space. Fix some $K \in D(X)$.
   (a) Give an example to show that the functor $U \mapsto \mathrm{Hom}(K|_U, K|_U)$ is not a sheaf in general.
   (b) Assume that $\mathrm{Ext}^i(K|_V, K|_V) = 0$ for $i < 0$ and $V \subset X$. Show that the functor $U \mapsto \mathrm{Hom}(K|_U, K|_U)$ is a sheaf.
   *(c) Let $\mathcal{C}_X$ be the full subcategory of $K \in D(X)$ satisfying the condition in (b) (and likewise for any open in X). Show that the association $U \mapsto \mathcal{C}_U$ is a *stack* on X, i.e., given an open cover $\{V_i\}$ of X, complexes $M_i \in \mathcal{C}_{V_i}$, and isomorphisms $M_i|_{V_i \cap V_j} \simeq M_j|_{V_i \cap V_j}$ for all i, j satisfying the usual cocycle condition over triple intersections, there is a unique $M \in \mathcal{C}_X$ inducing each M_i over V_i.

3. Let $\mathcal{A}$ be an abelian category with enough projectives. Assume that $\mathrm{Ext}^2(X, Y) = 0$ for any $X, Y \in \mathcal{A}$.
   (a) Show that for any $K \in D^b(\mathcal{A})$, there exists an isomorphism $K \simeq \oplus_i H^i(K)[-i]$.
   (b) Give an example to show the cohomology functor $H^* : D^b(\mathcal{A}) \to \mathcal{A}^{\mathbf{Z},b}$ is not an equivalence; here $\mathcal{A}^{\mathbf{Z},b}$ is the category of $\mathbf{Z}$-graded objects of $\mathcal{A}$, almost all of which are 0.

4. Classify all t-structures on $D^b_f(k)$ for a field k.

5. Let D be a triangulated category with a t-structure $(D^{\leq 0}, D^{\geq 0})$. If $K \in D$ is a retract (i.e., direct summand) of some $L \in D^{\leq 0}$, then show that $K \in D^{\leq 0}$ (and dually for $D^{\geq 0}$).

6. Let R be a dvr. Let $(D^{\leq 0}, D^{\geq 0})$ be some t-structure on $D := D^b_f(R)$. Assume that the t-structure is non-degenerate, i.e., $\cap D^{\leq n} = 0 = \cap D^{\geq n}$.
   (a) Assume that $R \in D^{\geq 0}$, but $R \notin D^{\geq 1}$. Using the truncation triangle
   $$ {}^pH^0(A) \to A \to {}^p\tau^{\geq 1}A, $$
   and exercises (3) and (5), conclude that $R \in D^{\heartsuit}$.
   (b) Let k denote the residue field of R. Show that for any i, one of the two maps ${}^p\tau^{\leq i}k \to k$ or $k \to {}^p\tau^{\geq i+1}k$ is an isomorphism.
   (c) In the situation of (a), show that the residue field k satisfies: $k \in D^{\heartsuit}$ or $k[-1] \in D^{\heartsuit}$.
   (d) Show that the functor $T(K) := R\mathrm{Hom}(K, R)$ is an autoduality of D, i.e., $T^2 = \mathrm{id}$. Calculate that $T(k) = k[-1]$.
   (e) Show that all t-structures on D can be obtained by hitting the standard t-structure on D with the following two operations: T or translations.

7. The goal of this exercise is to discuss examples of "glueing setups" coming from commutative algebra. Let R be a commutative ring containing an ideal I such that[1] I is flat, and $I^2 = I$.

   (a) Show that $R/I \otimes_R^L R/I \simeq R/I$ via the multiplication map.

   (b) Show that the restriction of scalars functor $i_* : D(R/I) \to D(R)$ is fully faithful.

   (c) Show that $K \in D(R)$ is in the image of i_* if and only if $I \otimes_R^L K \simeq 0$.

   (d) Show that i_* has a left-adjoint i^* and a right-adjoint $i^!$.

   Let D_U be the full subcategory of $D(R)$ spanned by complexes K such that the natural "action" map $I \otimes_R^L K \to K$ is an equivalence.

   (e) Show that D_U is a triangulated subcategory of $D(R)$. Let $j_! : D_U \to D(R)$ be the inclusion functor.

   (f) Show that $j_!$ has a right-adjoint j^* given by $M \mapsto I \otimes M$.

   (g) Show that j^* has a right-adjoint j_*, and that j_* is fully faithful.

   (h) Show that $D(R/I) \xrightarrow{i_*} D(R) \xrightarrow{j^*} D_U$ gives a "glueing setup" (as defined in class).

[1] Such objects tend to be non-noetherian. A typical example is $R = \cup_n \mathbf{C}[x^{\frac{1}{n}}]$ with $I = \cup_n (x^{\frac{1}{n}})$. More generally, one can show the following: if R is any *perfect* $\mathbf{F}_p$-algebra (such as the perfection of a noetherian ring), and $I \subset R$ is any radical ideal, then the pair (R, I) satisfies these assumptions; see *Projectivity of the Witt vector affine Grassmannian* on the arXiv.