Perverse sheaves: Problem set 2

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1. Let $\mathcal{D}$ be a small triangulated category. Define the K-group $K_0(\mathcal{D})$ as the free abelian group on the set of objects of $\mathcal{D}$ (write $[A]$ for the generator corresponding to some $A \in \mathcal{D}$) modulo the Euler relation, i.e., set $[A] = [B] + [C]$ if there is a short exact triangle

$$B \to A \to C \to B[1]$$

in $\mathcal{D}$.

   (a) Show that $[M[1]] = -[M]$, and conclude that each element of $K_0(\mathcal{D})$ is of the form $[A]$ for some $A \in \mathcal{D}$.

   (b) Show that giving a map $K_0(\mathcal{D}) \to G$ to an abelian group G is the same as specifying a map $f : |\mathcal{D}| \to G$ such that $f(A) = f(B) + f(C)$ for an exact triangle as above. (Here $|\mathcal{D}|$ denotes the set of objects of $\mathcal{D}$.)

   (c) Show that an exact functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ induces a homomorphism $K_0(\mathcal{D}_1) \to K_0(\mathcal{D}_2)$.

   (d) Recall that there is a K-group $K_0(\mathcal{A})$ associated to an abelian category $\mathcal{A}$. Show that $K_0(\mathcal{A}) \simeq K_0(D^b(\mathcal{A}))$.

   (e) Show that $K_0(D^b_{fg}(\mathrm{Ab})) = \mathbf{Z}$ and $K_0(D(\mathrm{Ab})) = 0$.

2. (Thomason) Let $\mathcal{D}$ be an essentially small triangulated category. A subcategory $\mathcal{C} \subset \mathcal{D}$ is called *admissible* if it is strict[1], full, triangulated, and dense (i.e., any object of $\mathcal{D}$ is a direct summand of an object of $\mathcal{C}$). For such $\mathcal{C}$, write $H_{\mathcal{C}} \subset K_0(\mathcal{D})$ for the image of the induced map $K_0(\mathcal{C}) \to K_0(\mathcal{D})$. Conversely, given a subgroup $H \subset K_0(\mathcal{D})$, write $\mathcal{C}_H$ for the full subcategory of $\mathcal{D}$ spanned by those $X \in \mathcal{D}$ with $[X] \in H \subset K_0(\mathcal{D})$. The goal of this problem is to show that the associations $H \mapsto \mathcal{C}_H$ and $\mathcal{C} \mapsto H_{\mathcal{C}}$ give a bijection between the set of subgroups of $K_0(\mathcal{D})$ and the set of admissible subcategories of $\mathcal{D}$; this is a theorem of Thomason.

   First, we fix a subgroup $H \subset K_0(\mathcal{D})$.

   (a) Show that $\mathcal{C}_H$ is admissible.

   (b) Show that $H_{\mathcal{C}_H} = H$.

   Now fix some admissible subcategory $\mathcal{C} \subset \mathcal{D}$. Define a relation $\sim$ on $|\mathcal{D}|$ by saying $X_1 \sim X_2$ if and only if there exist $A_1, A_2 \in \mathcal{C}$ such that $X_1 \oplus A_1 \simeq X_2 \oplus A_2$.

   (c) Show that $\sim$ is an equivalence relation on $|\mathcal{D}|$. Let G denote the associated quotient set, and write $\langle X \rangle \in G$ for the equivalence class of $X \in \mathcal{C}$.

   (d) Show that $X \in \mathcal{C}$ if and only if $\langle X \rangle = \langle 0 \rangle$.

   (e) Show that the direct sum operation endows G with the structure of an abelian monoid with $\langle 0 \rangle$ acting as the identity. Using the fact that $\mathcal{C}$ is dense, show that G is in fact a group.

   (f) Show that the Euler relation holds in G.

   (g) Show that there is a natural surjective homomorphism $K_0(\mathcal{D}) \to G$. Using (d), show that the kernel of this homomorphism is exactly $H_{\mathcal{C}}$. Conclude that $K_0(\mathcal{D})/H_{\mathcal{C}} \simeq G$.

   (h) Show that $\mathcal{C}_{H_{\mathcal{C}}} = \mathcal{C}$.

3. Let $\mathcal{A}$ be an abelian category. A complex $K \in K(\mathcal{A})$ is called K-*injective* if $\mathrm{Hom}_{K(\mathcal{A})}(C, K) = 0$ for any exact complex $C \in K(\mathcal{A})$.

[1] This means that every object of $\mathcal{D}$ isomorphic to an object of $\mathcal{C}$ is in $\mathcal{C}$.

(a) If K is K-injective, for any $M \in K(\mathcal{A})$, show that $\mathrm{Hom}_{K(\mathcal{A})}(M, K) \simeq \mathrm{Hom}_{D(\mathcal{A})}(M, K)$.

(b) Show that any bounded below complex with injective terms $\mathrm{Ch}(\mathcal{A})$ gives a K-injective complex in $K(\mathcal{A})$.

(c) Assume that for any $M \in K(\mathcal{A})$, there exists a quasi-isomorphism $M \to K$ with K being K-injective. Show that any left-exact functor $F : \mathcal{A} \to \mathcal{B}$ of abelian categories admits a right derived functor $RF : D(\mathcal{A}) \to D(\mathcal{B})$ defined by setting $RF(M) = F(K)$ in the previous notation.

(d) Show[2] that the hypothesis in (c) are satisfied by Grothendieck abelian categories.

4. Let $\mathcal{A}$ be a Grothendieck abelian category. Let I be a category

(a) If I is filtered, show that the colimit functor $\mathrm{colim} : \mathrm{Fun}(I, \mathcal{A}) \to \mathcal{A}$ is exact, and hence derives to a functor
$$\mathrm{hocolim} : D(\mathrm{Fun}(I, \mathcal{A})) \to D(\mathcal{A}).$$

(b) Show that the limit functor $\lim : \mathrm{Fun}(I^{\mathrm{opp}}, \mathcal{A}) \to \mathcal{A}$ is left-exact, and hence derives to a functor
$$\mathrm{holim} : D(\mathrm{Fun}(I^{\mathrm{opp}}, \mathcal{A})) \to D(\mathcal{A}).$$

(c) Construct products in $D(\mathcal{A})$ using (b).

*(d) Give an example where the products constructed in (c) do not coincide with the "naive" products.

(d) Let $I = \mathbf{N}$, so $\mathrm{Fun}(I^{\mathrm{opp}}, \mathcal{A})$ parametrizes projective systems $\{F_n\} := \{\dots F_{n+1} \to F_n \to \dots \to F_1\}$ of objects of $\mathcal{A}$. For any $\{K_n\} \in D(\mathrm{Fun}(I^{\mathrm{opp}}, \mathcal{A}))$, show that there is an exact triangle
$$\mathrm{holim}\, K_n \to \prod_n K_n \xrightarrow{a} \prod_n K_n \to \mathrm{holim}\, K_n[1]$$
where the map a is given by $t - \mathrm{id}$, where t denotes the transition map.

5. Let X be a topological space, and fix $x \in X$. Let J denote the poset of all open neighborhoods U of x.

(a) Show that the functor $\mathrm{Ab}(X) \to \mathrm{Ab}$ given by taking stalks at x derives to a functor $D(X) \to D(\mathrm{Ab})$; the latter is denoted $K \mapsto K_x$ in the sequel.

(b) For any $K \in D(X)$, construct a natural isomorphism $K_x \simeq \mathrm{hocolim}\, R\Gamma(U, K)$, where the colimit is indexed by J^{opp}.

6. Let $f : X \to Y$ be a map of topological spaces.

(a) Show that pullback $f^* : \mathrm{Ab}(Y) \to \mathrm{Ab}(X)$ is exact, and conclude that it derives to a functor $f^* : D(Y) \to D(X)$.

(b) Show that pushforward $f_* : \mathrm{Ab}(X) \to \mathrm{Ab}(Y)$ is exact, and conclude that it derives to a functor $Rf_* : D(X) \to D(Y)$.

(c) Show that Rf_* is right adjoint to f^*.

(d) For any $y \in Y$ and $F \in D(X)$, identify the stalk $(Rf_*F)_y \in D(\mathrm{Ab})$ as $\mathrm{hocolim}\, R\Gamma(f^{-1}U, F)$, where the colimit is indexed by open neighborhoods U of y.

7. Let $j : U \to X$ be an open immersion of topological spaces, and let $i : Z \to X$ be the closed complement.

(a) Show that j_* is fully faithful, its derived functor $Rj_* : D(U) \to D(X)$ is fully faithful, and that $j^* \circ Rj_* \simeq \mathrm{id}$ via the adjunction.

(b) Give an example to show that j_* is not exact.

(c) Show that extension by zero gives an exact functor $j_! : \mathrm{Ab}(U) \to \mathrm{Ab}(X)$, and conclude that it derives to a functor $j_! : D(U) \to D(X)$.

[2] Or, more realistically, read in the Stacks Project (Tag 079I).

(d) Show that $j_! : \mathrm{Ab}(U) \to \mathrm{Ab}(X)$ is fully faithful, and identifies $\mathrm{Ab}(U)$ with the collection of $F \in \mathrm{Ab}(X)$ such that $i^*F = 0$.

(e) Show that $j_! : D(U) \to D(X)$ is fully faithful, and identifies $D(U)$ with the collection of $K \in D(X)$ such that $i^*K = 0$.

(f) Show that $j_!$ is left adjoint to j^* (at both the abelian and derived levels). Conclude that $j^* \circ j_! \simeq \mathrm{id}$.

8. Let $i : Z \to X$ be a closed immersion of topological spaces, and let $j : U \to X$ be the complementary open.

(a) Show that $i_* : \mathrm{Ab}(Z) \to \mathrm{Ab}(X)$ is fully faithful, exact, and identifies $\mathrm{Ab}(Z)$ with the collection of $F \in \mathrm{Ab}(X)$ such that $j^*F = 0$.

(b) Show that $i_* : \mathrm{Ab}(Z) \to \mathrm{Ab}(X)$ admits a right adjoint $i^!$ given by sending $F \in \mathrm{Ab}(X)$ to the subsheaf $\underline{\Gamma}_Z(F) \subset F$ of "sections of F supported on Z" under the equivalence of (a).

(c) Show that $i^!$ derives to a functor $Ri^! : D(X) \to D(Z)$ that is right adjoint to $i_* : D(Z) \to D(X)$. Show also that the latter is fully faithful, and conclude that $Ri^! \circ i_* \simeq \mathrm{id}$.

(d) For any $F \in \mathrm{Ab}(X)$, construct exact sequences

$$0 \to i_* i^! F \to F \to j_* j^* F$$

and

$$0 \to j_! j^* F \to F \to i_* i^* F \to 0.$$

(e) Show that the first sequence above is right exact if F is injective. Give a counterexample without this hypothesis.

(f) Show that $i^* \circ j_! = 0$, $j^* \circ i_* = 0$, and $i^! \circ j_* = 0$. Give an example to show that $i^* \circ j_* \neq 0$. (All functors here are between the abelian categories of sheaves.)

(g) For any $K \in D(X)$, construct canonical exact triangles

$$i_* Ri^! K \to K \to Rj_* j^* K \to i_* Ri^! K[1]$$

and

$$j_! j^* K \to K \to i_* i^* K \to j_! j^* K[1].$$

(h) Show that $i^* \circ j_! = 0$, $j^* \circ i_* = 0$, and $Ri^! \circ Rj_* = 0$. Give an example to show that $i^* \circ Rj_* \neq 0$. (All functors here are between the derived categories of sheaves.)

9. Let X, Z, and U be as in the previous two problems. In each of the following cases, calculate $Rj_* \underline{\mathbf{Z}}$ and $Ri^! \underline{\mathbf{Z}}$.

(a) X a real manifold of dimension n, and $Z = \{x\}$ an isolated point on X.

(b) $X = S^1 \times S^1 / (S^1 \times \{x\})$ (the topological space underlying a nodal cubic), and $Z = \{x\}$ is the singular point.

(c) $X = M \times I / (M \times \{1\})$ is the cone on some manifold M, and $Z = \{x\}$ is the cone point.