Perverse sheaves: Problem set 1

Bhargav Bhatt

1. Fix a category $\mathcal{C}$.
   (a) Show that being additive is a property of $\mathcal{C}$, i.e., $\mathcal{C}$ supports at most one structure as an additive category.
   (b) Show the same as (a) above with "additive" replaced with "abelian."
2. Let $\mathcal{A}$ be an abelian category. Show that $\mathcal{A}$ admits all colimits if and only if it admits all direct sums.
3. Give examples of abelian categories satisfying:
   (a) Finite (resp. countable) direct sums exist, but countable (resp. all) direct sums do not.
   (b) Infinite products exist, but infinite direct sums do not.
   (c) Filtered colimits exist, but are not exact.
4. Is the category of torsion abelian groups a Grothendieck abelian category?
5. For any category I and Grothendieck abelian category $\mathcal{A}$, show that $\mathrm{Fun}(I, \mathcal{A})$ is Grothendieck abelian.
6. Let $i : \mathcal{A} \hookrightarrow \mathcal{B}$ be a fully faithful functor between abelian categories. Assume that $\mathcal{B}$ is Grothendieck abelian, and that i admits a left-adjoint L that is exact. Show that $\mathcal{A}$ is also Grothendieck. Using this criterion, check that the category of abelian sheaves on any site is Grothendieck abelian.
7. Verify that the category of chain complexes $\mathrm{Ch}(\mathcal{A})$ over an abelian category $\mathcal{A}$ is abelian. If $\mathcal{A}$ is Grothendieck, show that $\mathrm{Ch}(\mathcal{A})$ is Grothendieck as well.
8. Let $\mathcal{A} = \mathrm{Mod}_R$ for a ring R. Show that $\mathcal{A}$ admits a compact projective generator X, i.e., X is a generator, and the functor $\mathrm{Hom}(X, -)$ commutes with filtered colimits and is exact.
9. Let $\mathcal{A}$ be the category of $\mathbf{Z}$-graded vector spaces. Show that $\mathcal{A}$ is not equivalent to Mod_R for any ring R.
10. Let $\mathcal{A}$ be a Grothendieck abelian category that admits a compact projective generator X. Show that there is a natural equivalence $\mathcal{A} \simeq \mathrm{Mod}_R$ sending X to R, where $R = \mathrm{End}(X)$.
11. For any abelian category $\mathcal{A}$, write $\mathcal{A}^{\mathbf{N}}$ for the functor category $\mathrm{Fun}(\mathbf{N}^{\mathrm{opp}}, \mathcal{A})$ indexing projective systems $\{\dots X_{n+1} \to X_n \to \dots \to X_1\}$ in $\mathcal{A}$. Assume that $\mathcal{A}$ admits infinite direct sums and products.
    (a) Show that taking limits gives a left-exact functor $\lim : \mathcal{A}^{\mathbf{N}} \to \mathcal{A}$.
    (b) If infinite products are exact, show that the derived functors $\lim^i$ vanish for $i > 1$. Can you find a counterexample when infinite products are not exact?
    (c) Calculate the derived functors $\lim^i$ for the system $\{\dots p^{n+1}\mathbf{Z} \to p^n\mathbf{Z} \to \dots p\mathbf{Z} \to \mathbf{Z}\}$ with the maps being the obvious inclusion.
12. Fix an abelian category $\mathcal{A}$.
    (a) Verify that "homotopy-equivalence" is an equivalence relation on maps in $\mathrm{Ch}(\mathcal{A})$ compatible with composition.
    (b) Verify that homotopic maps in $\mathrm{Ch}(\mathcal{A})$ induce the same map on cohomology.

(c) Show that the homotopy category $K(\mathcal{A})$ is not abelian in general.

13. Let $f : K \to L$ be a map in $\mathrm{Ch}(\mathcal{A})$ for some abelian category $\mathcal{A}$.

    (a) Calculate the functor $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(\mathrm{cone}(f), -)$ in terms of f, K and L.

    (b) If L is acyclic, show that $H^i(\mathrm{cone}(f)) \simeq H^{i+1}(K)$.

    (c) If each $f^i : K^i \to L^i$ is split injective, show directly that the canonical map $\mathrm{cone}(f) \to L/K$ is a homotopy equivalence.

14. Let $\mathcal{A}$ be an abelian category, and let $\mathrm{Map}(\mathcal{A}) := \mathrm{Fun}([0 \to 1], \mathcal{A})$ be the category of maps in $\mathcal{A}$.

    (a) Show that $\mathrm{Map}(\mathcal{A})$ is abelian, and evaluating at either the source or the target gives exact functors $\mathrm{Map}(\mathcal{A}) \to \mathcal{A}$.

    (b) Show that the cone construction defines a functor $\mathrm{Ch}(\mathrm{Map}(\mathcal{A})) \to \mathrm{Ch}(\mathcal{A})$ that descends to a functor $K(\mathrm{Map}(\mathcal{A})) \to K(\mathcal{A})$.

    (c) Show that there is a canonical map $K(\mathrm{Map}(\mathcal{A})) \to \mathrm{Map}(K(\mathcal{A}))$. Give an example to show that this is not an equivalence in general.

    (d) Show that the construction in (b) does not factor to give a functor $\mathrm{Map}(K(\mathcal{A})) \to K(\mathcal{A})$ in general.

15. Identify the localization $\mathcal{C}[S^{-1}]$ for following pairs $\mathcal{C}, S$ in more familiar terms:

    (a) $\mathcal{C} = \mathrm{Ab}$, and S is the class of maps f such that $f \otimes \mathbf{Q}$ is an isomorphism.

    (b) $\mathcal{C} = \mathrm{Ab}$, and S is the class of maps f such that there exists some $n > 0$ with $n \cdot \ker(f) = n \cdot \mathrm{coker}(f) = 0$.

    (c) $\mathcal{C} = \mathrm{Ab}^{fg}$, and S is the class of maps f such that $f \otimes \mathbf{Z}/2$ is an isomorphism.

    (d) $\mathcal{C}$ is the category of schemes of finite type over a field k, and S is the class of universal homeomorphisms.

    (e) $\mathcal{C} = \mathrm{Ab}(X)$ for a topological space X with a given closed subspace Z, and S is the class of maps f such that $f|_Z$ is an isomorphism.

    (f) $\mathcal{C} = \mathrm{Coh}(X)$ for a projective variety X, and S is the class of maps f such that $H^0(X, f)$ is an isomorphism.

16. Prove the Gabriel-Zisman theorem.