PERFECTOID SPACES (MATH 679)

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Goal. The goal of this class is to introduce and study the theory of perfectoid spaces, and to discuss their applications to commutative algebra and algebraic geometry.

Background. Perfectoid spaces are a class of spaces in p-adic arithmetic geometry introduced in 2012 by Peter Scholze in his PhD thesis. Very roughly speaking, these spaces are modeled by (spectra attached to) $\mathbf{Z}_p$-algebras whose mod p reductions have all p-power roots. Despite their youth, these spaces have had stunning applications to many different areas of mathematics, including number theory, algebraic geometry, representation theory, commutative algebra, and algebraic topology. The key to this success is that perfectoid spaces provide a functorial procedure to translate certain algebro-geometric problems from characteristic 0 (or mixed characteristic) to characteristic p; the latter can often be more accessible thanks to the magic of Frobenius.

Plan. The bulk of this class will be devoted to explaining the basic theory of perfectoid spaces [Sc1, §1 - §7]. En route, we shall encounter the following topics:

- Faltings' theory of 'almost mathematics' (as explained in [GR]) conceived in his proof of Fontaine's conjectures in p-adic Hodge theory [Fa].
- Huber's approach to nonarchimedean geometry via his language of adic spaces [Hu1, Hu2, Hu3].
- The basic algebraic geometry of perfect schemes in characteristic p.

The highlight of this part of the course will be the 'almost purity theorem', a weaker version of which forms the cornerstone of Faltings' aforementioned work. The rest of the course will focus on the following applications:

- The direct summand conjecture and its derived version (following the ideas of [An, Bh]), and the p-adic analog of Kunz's theorem characterizing regularity.
- The Hodge-Tate decomposition for smooth projective varieties over p-adic fields (following [Fa, Sc2]).

Prerequisites. I will assume familiarity with the language of algebraic geometry (Math 631, Math 632). Prior experience with some form of rigid analytic geometry (say familiarity with the results discussed in [Bo] or [Sch], or participation in Mattias Jonsson's class on "Berkovich spaces" last fall) is not necessary, but will be very useful in motivating various constructions and results.

Homework. Some background material will be delegated to homework in the form of readings; I will also occasionally put up some problem sets at `http://www-personal.umich.edu/~bhattb/teaching/mat679w17/`

References

[An] Y. André, *La conjecture du facteur direct*, available at `https://arxiv.org/abs/1609.00345`.

[Bh] B. Bhatt, *The direct summand conjecture and its derived variant*, available at `https://arxiv.org/abs/1608.08882`

[Bo] S. Bosch, *Half a century of rigid analytic spaces*, Pure and Applied Mathematics Quarterly, Volume 5, Number 4.

[Fa] G. Faltings, *p-adic Hodge theory*, J. Amer. Math. Soc. 1, 1 (1988), 255-299.

[GR] O. Gabber and L. Ramero, *Almost ring theory*, Lecture Notes in Math. 1800, Springer (2003).

[Hu1] R. Huber. *Continuous valuations*, Math. Z., 212(3):455–477, 1993.

[Hu2] R. Huber. *A generalization of formal schemes and rigid analytic varieties*, Math. Z., 217(4):513–551,1994.

[Hu3] R. Huber, *Etale cohomology of rigid analytic varieties and adic spaces*, Aspects of Mathematics, E30. Friedr. Vieweg & Sohn, Braunschweig, 1996.

[Sch] P. Schneider, *Basic notions of rigid analytic geometry*, available at `http://www.math.ucsd.edu/~csorense/teaching/math299_adic/schneider_rigid.pdf`.

[Sc1] P. Scholze, *Perfectoid spaces*, Publ. Math. Inst. Hautes Études Sci. 116 (2012), 245-313.

[Sc2] P. Scholze, *p-adic Hodge theory for rigid-analytic varieties*, Forum Math. Pi 1 (2013), e1, 77.

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