PROBLEM SET 4 (MATH 631)

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The problems marked with (∗) do not need to be submitted. Some of them rely on material not covered yet in our class, and are meant to exploratory.

(1) Let Spec\( (B) \subset \text{Spec}(A) \) be an open immersion defined by a map \( A \to B \) of rings.

(a) Show that \( A \to B \) is flat.

(b) If \( A \to B \) is faithfully flat, conclude that \( A \cong B \).

(c) If \( R \) is an artinian local ring and \( A = R[x] \), show that \( B \cong A_f \) for some \( f \in A \).

(d) (∗) Is there an example where \( \text{Spec}(B) \) is not a distinguished open of \( \text{Spec}(A) \)?

(2) For any category \( C \), write \( PShv(C) \) for the category of presheaves on \( C \) (i.e., functors \( C^{\text{op}} \to \text{Sets} \)).

(a) (∗) Prove Yoneda’s lemma: the map \( X \mapsto h_X := \text{Hom}( -, X) \) extends to a fully faithful functor \( C \to PShv(C) \). A presheaf \( F \) is isomorphic to \( h_X \) for some \( X \in C \) is called \textit{representable}, and the object \( X \) (which is unique up to unique isomorphism) is called the representing object.

We now specialize to \( C = \text{Sch} \) being the category of schemes; write \( \text{Sch}^{\text{aff}} \) for the category of affine schemes. We call a presheaf \( F \) on \( \text{Sch} \) a \textit{sheaf} if it satisfies the sheaf axiom, i.e., if \( U \) is a scheme equipped with an open cover \( \{U_i\} \), then composition with each \( U_i \to U \) is required to induce a bijection

\[
F(U) \cong \text{Eq}( \prod_i F(U_i) \xrightarrow{s} \prod_{i,j} F(U_i \cap U_j) )
\]

where \( s \) and \( t \) are induced by the inclusions \( U_i \cap U_j \to U_i \) and \( U_i \cap U_j \to U_j \), and Eq refers to the equalizer\(^1\).

(b) For any \( X \in \text{Sch} \), show that \( h_X \) is a sheaf.

(c) Show that the full subcategory of sheaves inside \( PShv(\text{Sch}) \) and \( PShv(\text{Sch}^{\text{aff}}) \) are naturally identified. (For this reason, one often views \( h_X \) as a functor on affine schemes.)

If \( U = \text{Spec}(A) \) is an affine scheme and \( F \) is a presheaf, we often write \( F(A) = F(\text{Spec}(A)) \) if there is no possibility of confusion. For \( F = h_X \) and a commutative ring \( A \), we also simply write \( X(A) = h_X(\text{Spec}(A)) = \text{Hom}(\text{Spec}(A), X) \); this is called the \textit{functor of points} of \( X \).

(d) Decide if the following functors on \( \text{Sch} \) are representable. If representable, construct the corresponding representing objects; if not, explain why.

\(^1\)This is just the categorical name for the concept that showed up explicitly when we talked about sheaves. Concretely,

\[\text{Eq}( X \xrightarrow{s} Y ) = \{ x \in X \mid s(x) = t(x) \}.\]
(i) $U \mapsto \mathcal{O}(U)$
(ii) $U \mapsto \mathcal{O}(U) \otimes \mathcal{O}(U)$
(iii) $U \mapsto \mathcal{O}(U)^S$ for some set $S$.
(iv) $U \mapsto \{f, g \in \mathcal{O}_X(U) \mid f^2 = g^3\}$
(v) $U \mapsto \text{GL}_n(\mathcal{O}(U))$ (for fixed $n$)
(vi) $U \mapsto \{f, g \in \mathcal{O}_X(U)^2 \mid (f, g) = (1)\}$
(vii) $(\ast) U \mapsto \{f, g \in \mathcal{O}_X(U) \mid (f, g) = (1)\}/\mathcal{O}(U)^\ast$, where $\mathcal{O}(U)^\ast$ acts by scaling both factors.
(viii) $U \mapsto$ the set of open subsets of $U$

(e) Show that an affine scheme $X$ is locally of finite presentation over $\text{Spec}(\mathbb{Z})$ if and only if for any directed system $\{A_i\}$ of rings with limit $A_\infty := \varprojlim A_i$, we have
\[
\varprojlim_i X(A_i) \simeq X(A)
\]
via the natural map. (Bonus: show the same when $X$ is not assumed affine.)

(f) Give an example of a map $X \to Y$ of reduced schemes where $X(k) \simeq Y(k)$ for all fields $k$, but $X \to Y$ is not an isomorphism.

The following numbers refer to exercises in §II.3 of Hartshorne.

(3) 3.9
(4) 3.11
(5) 3.15
(6) 3.16
(7) 3.17
(8) 3.18
(9) 3.19. (Bonus: try to use Chevalley’s theorem to prove the following special case of the Ax-Grothendieck theorem: given a scheme $X$ of finite type over $\mathbb{Z}$ and an endomorphism $f : X \to X$, if the induced map $X(\mathbb{C}) \to X(\mathbb{C})$ is injective, it is also surjective.)
(10) 3.20
(11) 3.22