PROBLEM SET 11 (MATH 631)

BHARGAV BHATT

Problems marked with a (*) need not be submitted. All the numbered problems are from §II.8 in Hartshorne’s book.

1. Fix a field $k$, and a finite $k$-algebra $A$. Show that $\Omega^1_{A/k} = 0$ exactly when $A$ is a finite product of finite separable field extensions.

2. Say $A \to B$ is a map of rings. Assume $B$ is an $F_p$-algebra which is perfect (i.e., Frobenius is an isomorphism). Show that $\Omega^1_{B/A} = 0$.

3. Give examples of the following:
   
   (a) A finite field extension $L/K$ with $\Omega^1_{L/K} \neq 0$.
   
   (b) A non-algebraic extension $L/K$ with $\Omega^1_{L/K} = 0$.
   
   (c) A non-reduced $F_p$-algebra $R$ such that $\Omega^1_{R/F_p} = 0$.
   
   (d) A Dedekind scheme that is quasi-affine (i.e., occurs as a quasi-compact open subset of an affine scheme) but not affine.
   
   (e) A map $f : X \to Y$ of schemes whose diagonal $\Delta_f$ is an open immersion but not a closed immersion.

4. Let $X$ be a scheme, and let $V$ be a vector bundle on $X$. If $f : \text{Spec} \left( \text{Sym}^* V \right) \to X$ is the “geometric vector bundle” attached to $V$, describe $\Omega^1_f$ in terms of $V$.

5. Let $f : X \to \text{Bl}_0(\mathbb{A}^2)$ be the blowup at $0 \in \mathbb{A}^2$ over a field $k$. Describe $\Omega^1_f$ in terms of sheaves on the exceptional divisor $E \simeq \mathbb{P}^1$.

6. Let $X$ be a scheme, and $f : \mathbb{P}_X(V) \to X$ be the projectivization of a vector bundle $V$ on $X$. The goal of this exercise is to construct the Euler sequence

$$0 \to \Omega^1_f(-1) \to f^*V \to \mathcal{O}_{\mathbb{P}_X(V)}(1) \to 0$$

using the functor of points.

   (a) Fix a commutative ring $R$, a map $g : \text{Spec}(R) \to X$, and an invertible quotient $\alpha_g : g^*V \to L$, corresponding to a map $\overline{g} : \text{Spec}(R) \to \mathbb{P}_X(V)$ lifting $g$. Show that the collection of all $X$-maps

   $$\text{Spec}(R[\epsilon]/\epsilon^2) \to \mathbb{P}_X(V)$$

   lifting $\overline{g}$ over $\text{Spec}(R)$ identifies canonically with the cokernel of the map $R \to \text{Hom}(g^*V, L)$ sending $1$ to $\alpha_g$.

   (b) Use (1) to construct a short exact sequence

   $$0 \to \Omega^1_f(-1) \to f^*V \to \mathcal{O}_{\mathbb{P}_X(V)}(1) \to 0$$

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1A scheme is Dedekind if it is integral, noetherian, normal and has dimension $\leq 1$. 
of vector bundles on $\mathbb{P}_X(V)$.

(c) Give an example of such a pair $(X, V)$ with $X$ projective over a field $k$ and $\mathcal{O}_{\mathbb{P}_X(V)}(1)$ not ample as a line bundle on $\mathbb{P}_X(V)$.

(7) II.8.3

(8) (*) Please read about regular rings in a commutative algebra textbook. One suggestion: read pages 153 through 157 of Matsumura’s *Commutative Ring Theory*. 