Algebraic topology (Math 592): Problem set 7

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For any set \( S \), write \( \mathbb{Z}^\oplus S := \bigoplus_{s \in S} \mathbb{Z} \) for the free abelian group with basis \( S \). We view \( S \) as a subset of \( \mathbb{Z}^\oplus S \) in the evident way. Likewise, \( \mathbf{F}_p^\oplus S \) denotes the free \( \mathbf{F}_p \)-vector space with basis \( S \).

1. Give an example of a chain complex \( K \) of free abelian groups such that
   (a) \( H_0(K) = \mathbb{Z}/3 \), and \( H_1(K) = \mathbb{Z}/2 \), and the other \( H_i(K) \) vanish.
   (b) \( H_i(K) = \mathbb{Z}/2 \) for all \( i \geq 0 \).
   (c) \( H_0(K) = \mathbb{Q} \), and \( H_i(K) = 0 \) for \( i > 0 \).

2. Let \( G = (V, E) \) be a directed graph, i.e., \( V \) is a set, and \( E \subset V \times V \) is a subset; we view \( V \) as the set of vertices of \( G \), while an edge \( e = (x, y) \in E \) is viewed as an oriented path from \( x \) to \( y \). Define a two-term complex

\[
K(G) := (\mathbb{Z}^\oplus E \xrightarrow{d} \mathbb{Z}^\oplus V)
\]

with \( d \) uniquely characterized by \( d(e) = x - y \) for \( e = (x, y) \in E \subset \mathbb{Z}^\oplus E \). Assume \( G \) is finite.

   (a) Show that \( H_0(K(G)) = \mathbb{Z}^{\oplus \pi_0(G)} \).
   (b) Show that \( H_*(K(G)) \) only depends on the underlying undirected graph, i.e., if \( G' \) is obtained from \( G \) by replacing an edge \( e = (x, y) \) with \( e' = (y, x) \), then \( H_*(K(G)) \simeq H_*(K(G')) \).
   (c) Calculate \( H_1(K(G)) \) when \( V = \{x, y\} \), and \( E = V \times V \).
   (d) Can you guess a formula for \( H_1(K(G)) \) in general?

3. Let \( S \) be a non-empty set, and write \( S^i \) for the \( i \)-fold self-product\(^1\) of \( S \). Define a complex \( K_\bullet \) by setting \( K_i := \mathbf{F}_2^\oplus S^i \) for \( i \geq 0 \), and \( 0 \) if \( i < 0 \). The differential \( K_{i+1} \to K_i \) is the unique map sending \( (s_1, \ldots, s_{i+1}) \in S^{i+1} \subset \mathbf{F}_2^\oplus S^{i+1} \) to the sum of the \((i+1)\) elements of \( \mathbf{F}_2^\oplus S^i \) obtained by dropping exactly one co-ordinate\(^2\).

   (a) Show that the preceding recipe defines a chain complex (i.e., that \( d^2 = 0 \)).
   (b) Show that \( K \) is contractible, i.e., the identity on \( K \) is null-homotopic. (Hint: fix \( x \in S \), and use the maps \( h_i : K_i \to K_{i+1} \) determined by the map \( (s_1, \ldots, s_i) \mapsto (x, s_1, \ldots, s_i) \) on the basis elements \( S^i \subset K_i \).
   (c) Calculate \( H_*(K) \).

4. Let \( F : \text{Ab} \to \text{Ab} \) be an additive functor, i.e., \( F \) is a functor such that the induced map \( \text{Hom}(M, N) \to \text{Hom}(F(M), F(N)) \) is a group homomorphism for \( M, N \in \text{Ab} \).

   (a) Show that \( F \) naturally induces a functor \( \text{Ch}(\text{Ab}) \to \text{Ch}(\text{Ab}) \) by “termwise application.”
   (b) Consider the additive functor \( F(M) = M/2M \). Show that \( F \) does not carry short exact sequences of chain complexes to short exact sequences.
   (c) Consider the additive functor \( F(M) = M_{\text{tors}} \) (i.e., the torsion subgroup of \( M \)). Give an example of an acyclic complex \( K \in \text{Ch}(\text{Ab}) \) such that \( F(M) \) is not acyclic.

\(^1\)For \( i = 0 \), we have the empty product in the category of sets. The empty product is the final object in any category, provided it exists. So, in this case, \( S^0 \) is the set with 1 element; if you do not like this reasoning, take this as a definition.

\(^2\)For example, the differential \( \mathbf{F}_2^\oplus S^2 \to \mathbf{F}_2^\oplus S^2 \) sends a basis element \( (x, y, z) \in S^3 \subset \mathbf{F}_2^\oplus S^3 \) to \((x, y) + (y, z) + (x, z) \in \mathbf{F}_2^\oplus S^2 \).
(d) Give a natural example of a non-additive functor $G : \text{Ab} \to \text{Ab}$.

5. Give an example of a map $f : K \to L$ of chain complexes of abelian groups such that $H_*(f) = 0$, but $f$ is not null-homotopic. Can this happen with chain complexes of vector spaces over a field? Either prove not, or give an example.

6. Let $0 \to K \to L \to M \to 0$ be a short exact sequence of chain complexes of vector spaces over a field. Is the boundary map $H^i(M) \to H^{i+1}(K)$ necessarily 0?

7. Let $f : K \to L$ be a map of bounded chain complexes of free abelian groups. Assume that $H_*(f)$ is an isomorphism. Prove that $f$ is a homotopy equivalence.

8. Fix an integer $p$. Given a chain complex $K$, define $K[p]$ to be the shift of $K$ to the left, i.e., $K[p]_i = K_{i-p}$ with differential $d_{K[p]}$ coinciding with $(-1)^pd_K$. Show that $K[p]$ is a chain complex, and $K \mapsto K[p]$ is an autoequivalence of the category of chain complexes. What is the inverse?

9. Let $f : K \to L$ be a morphism of chain complexes. Define a new chain complex $C(f)$ via $C(f)_i = L_i \oplus K_{i-1}$ with differential $d(\ell,k) = (d(\ell) - f(k), -d(k))$.
   
   (a) Show that $C(f)$ is a chain complex.
   
   (b) Construct a natural short exact sequence

   $$0 \to L \to C(f) \to K[1] \to 0$$

   of chain complexes. Identify the boundary map in terms of the original map.

   (c) Show that $H_*(f)$ is an isomorphism exactly when $C(f)$ is acyclic (i.e., $H_*(C) = 0$).

   (d) Let $\alpha : C(f) \to K[1]$ be the map constructed above. Show that there is a natural homotopy equivalence $C(\alpha) \simeq L[1]$.

   (e) Consider the case $K = L$ with $f = \text{id}$. Show that $C(f)$ is contractible (i.e., identity is map is null-homotopic).

10. Given chain complexes $K$ and $L$, prove carefully that homotopy gives an equivalence relation on the set $\text{Hom}(K, L)$ of all chain complex maps $K \to L$. Check that this equivalence relation is compatible with composition and addition, and deduce that the category whose objects are chain complexes and whose morphisms are given by homotopy classes of chain complex maps is an additive category; this is often called the homotopy category of chain complexes. You do not need to submit this problem.