Algebraic topology (Math 592): Problem set 6

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All spaces are assumed to be Hausdorff and locally path-connected, i.e., there exists a basis of path-connected open subsets of $X$.

1. Let $f : X \to Y$ be a covering space. Assume that there exists $s : Y \to X$ such that $f \circ s = \text{id}$. Show that $s(Y) \subset X$ is clopen. Moreover, also check that $(X - s(Y)) \to Y$ is a covering space, provided $X \neq s(Y)$ and $Y$ is path-connected.

2. Let $\alpha : Z \to Y$ be a map of spaces. Fix a base point $z \in Z$, and assume that $Z$ and $Y$ are path-connected, and that $Y$ admits a universal cover. Show that $\alpha_* : \pi_1(Z, z) \to \pi_1(Y, \alpha(z))$ is surjective if and only if for every connected covering space $X \to Y$, the pullback $X \times_Y Z \to Z$ is connected.

All spaces are now further assumed to be path-connected unless otherwise specified.

3. Let $G$ be a topological group, and let $p : H \to G$ be a covering space. Fix a base point $h \in H$ living over the identity element $e \in G$. Using the lifting property, show that $H$ admits a unique topological group structure with identity element $h$ such that $p$ is a group homomorphism. Using a previous problem set, show that the kernel of $p$ is abelian.

4. Let $f : X \to Y$ be a covering space. Assume that there exists $y \in Y$ such that $\#f^{-1}(y) = d$ is finite. Show that $\#f^{-1}(y') = d$ for all $y' \in Y$. This number is called the degree of a covering space.

Recall that the group of deck transformations of a covering space $f : X \to Y$ is defined as $\text{Aut}_Y(X) := \{ \psi \in \text{Aut}(X) \mid f \circ \psi = f \}$. This group naturally acts on the fibres of the map $f$.

5. Given a covering space $f : X \to Y$ and a base point $y \in Y$, show that $\text{Aut}_Y(X)$ acts freely (i.e., without stabilizers) on $f^{-1}(y)$.

In the above situation, we say that $f$ is regular or Galois if $\text{Aut}_Y(X)$ acts transitively on $f^{-1}(y)$.

6. Show that universal covers (if they exist) are Galois.

7. Let $f : X \to Y$ be a Galois covering space. Show that $\text{Aut}_Y(X)$ acts properly discontinuously on $X$, and that $X/\text{Aut}_Y(X) \cong Y$ via $f$.

8. Let $f$ be a covering space, and fix $x \in X$. Show that $f$ is Galois if and only if $f_* (\pi_1(X, x)) \subset \pi_1(Y, f(x))$ is a normal subgroup. Deduce that if $\pi_1(Y)$ is abelian, any covering space is Galois.

9. Show that all covering spaces of degree 2 are Galois.

10. For any covering space $f : Y \to X$ of degree $n$, show that there exists a Galois covering space $g : Z \to X$ of degree dividing $n!$ and a map $Z \to Y$ of covering spaces. (You may assume $X$ admits a universal cover.)

11. Give an example of a covering space of $S^1 \vee S^1$ of degree 3 that is not Galois. (Hint: construct a non-normal index 3 subgroup of $\mathbb{Z} \ast \mathbb{Z}$ using the symmetric group $S_3$.)

12. Let $X = S^1 \vee S^1$. Count the number of connected covering spaces $Y \to X$ of degree 3 up to equivalence.

13. Does the group $\{ x \in \mathbb{Q} \mid 2^n x \in \mathbb{Z} \forall n \gg 0 \}$ arise as the fundamental group of a covering space of $S^1 \vee S^1$?
14. Let $X = S^1 \vee S^1$. Choose a map $f : S^1 \to S^1 \vee S^1$ representing an element $\alpha \in \pi_1(S^1 \vee S^1)$ that is well-defined up to conjugation. Construct the space $Y$ be glueing a two-dimensional closed unit disc $D^2$ to $X$ using the map $f$ on the boundary $\partial D^2 = S^1$, i.e., $Y$ is the pushout of $D^2 \leftarrow \partial D^2 \cong S^1 \xrightarrow{f} X$. Determine $\pi_1(D)$.

15. Using the construction of the previous exercise, show that any finite group $G$ arises as the fundamental group of a space. (Please attempt this even if you already tried this on the previous problem set.)