Algebraic topology (Math 592): Problem set 2

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1. Consider the circle $S^1$.
   (a) Let $f : S^1 \to S^1$. Given $x \in S^1$, we obtain an induced map $f_* : \mathbb{Z} \to \mathbb{Z}$, where the first $\mathbb{Z}$ is $\pi_1(S^1, x)$, while the second $\mathbb{Z}$ is $\pi_1(S^1, f(x))$. This map is uniquely determined by the integer $f_*(1)$, called the degree $\deg(f)$. Show that this integer is independent of $x$.
   (b) Given two maps $f, g : S^1 \to S^1$, show that $\deg(g \circ f) = \deg(g) \cdot \deg(f)$.
   (c) Show that if $f : S^1 \to S^1$ is not surjective, then $\deg(f) = 0$.
   (d) Show that any $f : S^1 \to S^1$ with $\deg(f) \neq 1$ must have a fixed point.
   (e) Construct a surjective map $f : S^1 \to S^1$ with $\deg(f) = 0$.

2. Let $X_n = \mathbb{R}^n - \{0\}$, and let $i_n : X_n \to X_{n+1}$ be the inclusion allowing us view $X_n$ as a subspace of $X_{n+1}$ of vectors with the last co-ordinate $0$. We point these spaces using $x_n := (1, 0, 0, \ldots, 0) \in X_n$.
   (a) For any loop $\alpha$ in $\pi_1(X_n, x_n)$, show that the induced loop $i_{n, *}(\alpha) \in \pi_1(X_{n+1}, x_{n+1})$ is $0$.
   (b) Let $X_\infty = \cup_n X_n$, given the colimit (or “weak”) topology. For any compact Hausdorff space $Y$, show that any map $Y \to X_\infty$ factors through some $X_n$.
   (c) Show that any map $S^n \to X_\infty$ is null-homotopic for any $n \geq 0$. Conclude that $\pi_1(X_\infty, x_\infty) = 0$, where $x_\infty = (1, 0, 0, 0, 0 \ldots)$.
   We will see later that $\pi_1(X_n, x_n) = 0$ for $n \geq 3$.

3. We prove that fundamental groups of topological groups are abelian.
   (a) Let $S$ be a set equipped with two binary operations $*$ and $\otimes$. Assume that there exist units $e_*$ and $e_\otimes$ for each operation, and that $*$ : $S \times S \to S$ is a homomorphism with respect to $\otimes$, i.e., $(a \otimes b) * (c \otimes d) = (a * c) \otimes (b * d)$. Show that $* = \otimes$, and both are commutative.
   (b) Let $G$ be a topological group with identity $e$. Show that $\pi_1(G, e)$ is abelian.

4. We will check that all maps $S^n \to S^1$ are null-homotopic. Let $\exp : \mathbb{R} \to S^1$ be the exponential $t \mapsto e^{2\pi it}$.
   (a) Let $D^o \subset \mathbb{R}^n$ be the open unit disc. Imitating the proof of $\pi_1(S^1, 1) \simeq \mathbb{Z}$, show that any map $f : D^o \to S^1$ lifts to a map $g : D^o \to \mathbb{R}$. Classify all possible such lifts $g$ of a given $f$.
   (b) Let $(X, x)$ be a pointed space such that $X = U_1 \cup U_2$, where each $U_i$ is homeomorphic to an open unit disc in $\mathbb{R}^n$, and $U_1 \cap U_2$ is path-connected. Show that for any map $f : X \to S^1$ and fixed $t_0 \in \mathbb{R}$ such that $\exp(t_0) = f(x)$, there is a unique map $\tilde{f} : X \to \mathbb{R}$ such that $\tilde{f}(x) = t_0$, and $\exp \circ \tilde{f} = f$. 

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(c) Show that any map \( S^n \to S^1 \) is nullhomotopic for \( n \geq 2 \).

5. Given pointed spaces \((X, x)\) and \((Y, y)\), show that \( \pi_1(X, x) \times \pi_1(Y, y) \simeq \pi_1(X \times Y, (x, y)) \).

EDIT: This problem was already asked on the first problem set, so please ignore it.

6. We prove that \( \pi_1(S^n) = 0 \) for \( n \geq 2 \). Fix a point \( x_0 \in S^n \) with antipode \( x_0 \). Let \( U = S^n - \{x_0\} \), and \( V = S^n - \{x_0\} \).

(a) Show that \( U \) and \( V \) are homeomorphic to \( \mathbb{R}^n \), and \( U \cap V \) is homeomorphic to \( \mathbb{R}^n - \{0\} \).

(b) Show that if \( m \gg 0 \), then each \( \alpha_j \) has image contained entirely in one of \( U \) or \( V \).

(c) Show that after repeatedly applying the following operations — replace some \( \alpha_i \) by a homotopic path, and compose adjacent paths in the factorisation \( \alpha = \alpha_{m-1} \cdot \alpha_{m-2} \cdot \ldots \cdot \alpha_0 \) of \( \alpha \) — we obtain “factorisation” \( \alpha = \alpha_{m-1} \cdot \alpha_{m-2} \cdot \ldots \cdot \alpha_0 \) of \( \alpha \) as a composition of \( m \) paths.

(d) Show that \( \alpha \) is null-homotopic, and conclude that \( \pi_1(S^n, x_0) = 0 \).

7. Assume there exists some space \((X, x)\) such that \( \pi_1(X, x) \) is not abelian. Show that \( \pi_1(S^1 \vee S^1, 1) \) is not abelian. Here recall that for pointed spaces \((X, x)\) and \((Y, y)\), we write \( X \vee Y := (X \sqcup Y) / (x \sim y) \).

8. Consider the groupoid \( BG \) for a finite group \( G \).

(a) Show that the category \( \text{Fun}(BG, \text{Ab}) \) of functors \( BG \to \text{Ab} \) is naturally equivalent to the category of abelian groups equipped with a \( G \)-action.

(b) Given a functor \( F : BG \to \text{Ab} \) corresponding to an abelian group \( M \) with a \( G \)-action, describe the limit and colimit of \( F \) in terms of \( M \).

9. Let \( G \) and \( H \) be two finite groups. Describe the category \( \text{Fun}(BG, BH) \) of functors \( BG \to BH \) explicitly (i.e., describe isomorphisms classes of objects as well as the automorphism group of each object).

10. Consider the category of algebraic closures of a field \( k \), i.e., the objects are algebraic extensions \( L/k \) with \( L \) algebraically closed, and the morphisms are \( k \)-algebra maps. Is this category a groupoid? If so, can you describe it in terms of the absolute Galois group of \( k \)?

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1The objects of this category are functors \( F : BG \to \text{Ab} \); morphisms are given by natural transformations, and composition is given by composition of natural transformations.