Algebraic topology (Math 592): Problem set 13
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1. Let $X$ be a space such that $H_*(X; \mathbb{F}_p) = 0$ for all primes $p$ and $H_*(X; \mathbb{Q}) = 0$. Show that $H_*(X) = 0$.

2. Fix a space $X$ and a commutative ring $k$.
   (a) There is a canonical map $\text{Hom}(M, \mathbb{Z}) \otimes k \to \text{Hom}(M, k)$ for any abelian group $M$ given by $f \otimes \alpha \mapsto (m \mapsto \alpha f(m))$. Using this, construct a natural map $H^*(X) \otimes k \to H^*(X; k)$ which is an isomorphism when $X = \text{pt}$.
   (b) Give an example of a space $X$ such that the map in (1) is not an isomorphism for $k = \mathbb{F}_p$.
   (c) Show that if $X$ is a finite CW complex, then the map in (1) is an isomorphism for $k = \mathbb{Q}$.
   (d) Give an example of a space $X$ such that the map in (1) is not an isomorphism for $k = \mathbb{Q}$.

3. Let $(X, x)$ and $(Y, y)$ be pointed spaces. Show that $H^*(X \vee Y)$ is identified with the subring of $H^*(X) \times H^*(Y)$ determined by pairs $(a, b)$ such that $i_x^* a = i_y^* b$, where $i_x : \{x\} \hookrightarrow X$, and $i_y : \{y\} \hookrightarrow Y$ are the inclusions.

4. Using the description of the graded ring $H^*(\mathbb{RP}^n; \mathbb{Z}/2)$ as input, describe $H^*(\mathbb{RP}^n)$ as a graded ring.

5. Show that there is no map $\mathbb{CP}^2 \to S^2$ that induces a nonzero map on $H_2$.

6. Show that any endomorphism of $\mathbb{CP}^n$ has a fixed point.

7. (Interpreting Ext-groups as extensions) Let $\mathcal{A} = \text{Mod}_R$ be the category of $R$-modules for a commutative ring $R$. Fix $X, Y \in \mathcal{A}$, and $n \geq 1$. A degree $n$ Yoneda extension of $X$ by $Y$ is an exact sequence
   $Z_* := 0 \to Y \to Z_1 \to \cdots \to Z_n \to X \to 0$.
   A map $Z_* \to Z'_*$ of such extensions is a map of exact sequences which is the identity on the $Y$ and $X$ terms. Two such extensions $Z'_*$ and $Z''_*$ are declared to be equivalent if there are maps $Z'_* \hookrightarrow Z_* \twoheadrightarrow Z''_*$ of extensions.
   (a) Show that equivalence of extensions is an equivalence relation on the set of all degree $n$ Yoneda extensions of $X$ by $Y$. The quotient set is denoted $\text{Ext}^n_{\mathcal{A}}(X, Y)$.
   (b) Show that $\text{Ext}^n_{\mathcal{A}}(X, Y)$ is covariantly functorial in $Y$, and contravariantly functorial in $X$ by considering pushouts and pullbacks of extensions.
   (c) Show that there is a natural binary operation $+_{\mathcal{A}}$ on $\text{Ext}^n_{\mathcal{A}}(X, Y)$ given by setting $[Z_*] + [Z'_*]$ to be the degree $n$ extension obtained by taking the direct sum $W_* := Z_* \oplus Z'_*$, which is an element in $\text{Ext}^n_{\mathcal{A}}(X \oplus Y, Y \oplus Y)$, and composing with the “fold” map $Y \oplus Y \to Y$ and the diagonal map $X \to X \oplus X$.
   (d) Let $e_{X,Y}$ be the degree $n$ extension obtained as follows: $Z_1 = Y$, $Z_n = X$ and $Z_i = 0$ for $i \neq 1, n$ if $n \geq 2$, and $Z_1 = X \oplus Y$ if $n = 1$ (and the maps are the obvious ones in both cases). Show that $e_{X,Y}$ is a unit for the operation $+_{\mathcal{A}}$ defined above.
   (e) By tweaking signs, show that $\text{Ext}^n_{\mathcal{A}}(X, Y)$ is an abelian group under $+$.
   (f) For $X, Y, W \in \mathcal{A}$, and $m, n \in \mathbb{Z}_{\geq 0}$, construct a natural map $\text{Ext}^n_{\mathcal{A}}(X, Y) \times \text{Ext}^m_{\mathcal{A}}(Y, W) \to \text{Ext}^{n+m}_{\mathcal{A}}(X, W)$ by splicing extensions together. Show that this operation is bilinear with respect to $+$, and associative.
   (g) (For those who know more homological algebra) Show that $\text{Ext}^n_{\mathcal{A}}(X, -) = 0$ for all $n \geq 1$ if and only if $X$ is projective. Dually, show that $\text{Ext}^n_{\mathcal{A}}(-, Y) = 0$ for all $n \geq 1$ if and only if $Y$ is injective.