1. Fix some \( n > 0 \), and let \( \deg : \text{Map}(S^n, S^n) \to \mathbb{Z} \) be the map defined by setting \( \deg(f) \) to be the unique integer such that \( H_n(S^n) \xrightarrow{H_n(f)} H_n(S^n) \) is multiplication by \( \deg(f) \). Show that \( \deg \) is surjective.

2. Find an example of a finite CW complex \( X \) with \( \chi(X) \neq 0 \) and a fixed point free endomorphism \( f : X \to X \).

3. Let \( X \) be a finite CW complex with \( n \)-skeleton \( X^n \) for each \( n \). Assume \( X^0 = \{ \ast \} \) for simplicity.
   (a) Show that if \( \pi_1(X)^{ab} \) is finite, then all maps \( X \to S^1 \) are null-homotopic.
   (b) Arguing inductively along cell attachments, show that every map \( X^2 \to S^1 \) extends to a map \( X \to S^1 \).
   (c) Show that any map \( \pi_1(X^1) \to \mathbb{Z} \) of groups can be realized as the map on fundamental groups induced by a map \( X^1 \to S^1 \) of spaces.
   (d) Show that any map \( \pi_1(X^2) \to \mathbb{Z} \) of groups can be realized as the map on fundamental groups induced by a map \( X^2 \to S^1 \) of spaces.
   (e) Show that if all maps \( X \to S^1 \) are null-homotopic, then \( \pi_1(X)^{ab} \) is finite.

4. Let \( f : X \to Y \) be a Galois covering space of degree \( d \) with \( X \) path-connected. Let \( G \) be the covering group.
   (a) Given a finite CW complex structure on \( Y \), construct one on \( X \) to make \( f \) a cellular map. Your construction should have the property that the \( G \)-action permutes all cells of \( X \) lying above a given cell of \( Y \) in a simply transitive fashion.
   (b) Given a finite CW complex structure on \( X \) that is \( G \)-equivariant (i.e., the \( G \)-action on \( X \) carries cells to cells compatibly with the attaching maps), construct a CW structure on \( Y \) making \( f \) a cellular map.
   (c) Show that the constructions in (a) and (b) are mutually inverse, i.e., finite CW structures on \( Y \) identify with \( G \)-equivariant finite CW structures on \( X \).

5. Let \( X \) be a space, and let \( \{U_1, \ldots, U_n\} \) be an open cover of \( X \). For each subset \( S \subset \{1, \ldots, n\} \), write \( U_S = \bigcap_{s \in S} U_s \). Assume that for each nonempty \( S \), the group \( \oplus_i H_i(U_S) \) is finitely generated. Write down a formula computing \( \chi(X) \) in terms of the \( \chi(U_S) \) for \( S \subset \{1, \ldots, n\} \) nonempty. (Hint: think about \( n = 2 \) first.)

6. Let \( f : X \to Y \) be a covering space of degree \( d \). Assume that \( Y \) admits a good cover, i.e., an open cover \( \{U_1, \ldots, U_n\} \) such that each \( U_S \) is contractible for \( \emptyset \neq S \subset \{1, \ldots, n\} \), \( \emptyset \subset \{1, \ldots, n\} \) nonempty. (Every compact smooth manifold admits such a cover, see Bott-Tu.)
   (a) Show that \( \oplus_i H_i(X) \) and \( \oplus_i H_i(Y) \) are finitely generated abelian groups.
   (b) Show that the rank\(^1\) of \( H_i(Y) \) is bounded above by that of \( H_i(X) \).
   (c) Show that \( \chi(X) = \chi(Y) \cdot d \).
   (d) Give an example of such a map \( f \) such that \( H_2(f; \mathbb{F}_2) \) is the 0 map between nonzero vector spaces.

7. This problem concerns coverings of surfaces. You may assume the existence of good covers.
   (a) Show that there is no non-trivial covering space \( \Sigma_g \to \Sigma_g \) for \( g \neq 1 \) (where \( \Sigma_g \) is the compact oriented genus \( g \) surface).

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\(^1\)If \( A \) is a finitely generated abelian group, then \( A \simeq \mathbb{Z}^r \oplus T \) where \( T \) is torsion. The integer \( r \) is called the rank.
(b) If $X \to \Sigma_g$ is a finite degree covering space with $X$ connected, show that $X \simeq \Sigma_h$ for suitable $h$, and calculate $h$.

(c) Describe all non-trivial finite degree covers $\Sigma_1 \to \Sigma_1$.

8. Fix a fixed abelian group $A$, check carefully that $A \otimes -$ commutes with all colimits.

9. If $V$ and $W$ are modules over some ring $R$, then they can also be viewed as abelian groups. Construct a natural map $V \otimes_Z W \to V \otimes_R W$. For which of the following rings $R$ is this map always an isomorphism:

(a) $R = \mathbb{Z}/n$.
(b) $R = \mathbb{Z}/n \times \mathbb{Z}/m$.
(c) $R = \mathbb{Z}[\frac{1}{n}]$.
(d) $R = \mathbb{Q}$.
(e) $R = \mathbb{R}$.

10. Fix an abelian group $M$. Are the following true or false? Justify your answer:

(a) If $M$ is finitely generated, and $M \otimes \mathbb{Z}/n = 0$ for all $n > 0$, then $M = 0$.
(b) If $M \otimes \mathbb{Z}/n = 0$ for all $n > 0$, then $M = 0$.
(c) If $M \otimes \mathbb{Z}/n = 0$ for all $n > 0$, and $M \otimes \mathbb{Q} = 0$, then $M = 0$.
(d) If $M \otimes M \simeq M$, then $M = 0$.
(e) If $M \otimes M \simeq 0$, then $M = 0$.
(f) If $M \otimes \mathbb{Z}/p \simeq 0$, then $M \otimes \mathbb{Z}/p^k = 0$ for all $k > 0$.
(g) If $M \otimes \mathbb{Q} \simeq \mathbb{Q}$, and $M \otimes \mathbb{Z}/p \simeq \mathbb{Z}/p$ for all primes $p$, then $M \simeq \mathbb{Z}$. 

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