This is an 80-minute exam, but you have the full 110-minute class period to complete it. There are a total of 100 points possible; the value of each problem is marked. Partial credit may be given for significant progress toward the solution to a problem. Except where otherwise indicated, you must show sufficient work to make it clear to me what you are doing and why in order to obtain full credit.

Problem 1 (10 pts)
Compute the determinant of the following matrix by whatever method seems best to you.

\[
\begin{pmatrix}
  1 & 2 & 0 & 0 \\
  4 & -1 & 4 & 1 \\
  3 & 5 & 0 & 0 \\
 -7 & 3 & -1 & 2
\end{pmatrix}
\]

Problem 2 (10 pts)
The set of points in \(xyz\)-space with \(x + y + z = 0\) forms a plane \(V\). Find the matrix which represents orthogonal projection onto \(V\).

Problem 3 (10 pts)
Consider the following matrix.

\[
A = \begin{pmatrix}
  2 & -1 & 2 \\
  1 & 3 & 5 \\
  2 & 0 & 3
\end{pmatrix}
\]

a) Compute \(\text{adj } A\).

b) It turns out that \(\det A = -1\). (You don’t have to verify this fact.) Compute \(A^{-1}\).

Problem 4 (10 pts)
Consider the following linear system.

\[
\begin{align*}
2x + y &= 5 \\
x - y &= 3 \\
x &= 3
\end{align*}
\]

Write down the corresponding normal equation. (You don’t need to solve it.)

Problem 5 (10 pts)
Fill in the missing entries so that the matrix is orthogonal.

\[
\begin{pmatrix}
  3/13 & 4/13 \\
  4/13 & -12/13 \\
 12/13 & -4/13
\end{pmatrix}
\]
Problem 6 (15 pts)

Let $V$ be the subspace of $\mathbb{R}^5$ with a basis $A = \left\{ \begin{pmatrix} 2 & 7 \\ 2 & 1 \\ 2 & 1 \\ 3 & 6 \end{pmatrix}, \begin{pmatrix} 7 & 1 \\ 2 & 7 \\ 2 & 1 \end{pmatrix} \right\}$.

a) Compute the QR decomposition of \( \begin{pmatrix} 2 & 7 \\ 2 & 1 \\ 2 & 1 \\ 3 & 6 \end{pmatrix} \).

b) Write down an orthonormal basis $B$ of $V$.

c) Consider the vectors $\vec{v}$, $\vec{w}$ whose $A$-coordinates are $\left( \frac{2}{3} \right)$ and $\left( \frac{1}{-1} \right)$. Express these vectors in $B$-coordinates.

d) What is the cosine of the angle between $\vec{v}$ and $\vec{w}$?

Problem 7 (15 pts)

Consider the linear transformation $T: P_2 \to P_2$ defined by $T(f(t)) = f(2t + 1) - f(t + 1)$.

a) Write the matrix representation of $T$, relative to the basis $\{t^2, t, 1\}$.

b) Write down a basis for the kernel of $T$.

c) Write down a basis for the image of $T$.

d) What is the nullity (the dimension of the kernel) of $T$?

e) What is the rank (the dimension of the image) of $T$?

f) What is the determinant of $T$?

Problem 8 (20 pts)

Indicate whether each item is true or false. Each item is worth 2 points. No explanation necessary, no partial credit possible.

a) T / F ? If a $2 \times 2$ matrix $P$ represents the orthogonal projection onto a line in $\mathbb{R}^2$, then $P$ is similar to $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

b) T / F ? There exists an isomorphism from $P_4$ to the space of $2 \times 2$ matrices.

c) T / F ? The invertible $5 \times 5$ matrices form a subspace of the space of $5 \times 5$ matrices.

d) T / F ? If $A$ and $B$ are any linear transformations from a linear space $V$ to itself, then ker($AB$) contains ker $B$.

e) T / F ? A matrix is orthogonal if and only if it has orthogonal columns.

f) T / F ? Every system of linear equations has a (not-necessarily unique) least-squares solution.

g) T / F ? If $A$, $B$, $C$ are symmetric $n \times n$ matrices, then $ABACABA$ is also a symmetric matrix.

h) T / F ? The matrices $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ and $\begin{pmatrix} 3 & -4 \\ 4 & 6 \end{pmatrix}$ are similar.

i) T / F ? If $(v_1 \ v_2 \ v_3 \ v_4)$ is a $4 \times 4$ matrix with determinant 10, then the matrix $(v_1 \ 2v_2 \ v_1 \ v_1 + 3v_2)$ has determinant $-60$.

j) T / F ? A $3 \times 3$ matrix has determinant zero if and only if its rows are linearly dependent.