Problem 1 (10 points):

Find the equation of the cubic equation (general form: \( f(x) = ax^3 + bx^2 + cx + d \)) whose graph passes through the points \((-1, 2), (0, 3), (1, 6), (2, 23)\).

Answer.

These four points lead to the following linear system.

\[
\begin{align*}
-a + b - c + d &= 2 \\
d &= 3 \\
a + b + c + d &= 6 \\
8a + 4b + 2c + d &= 23
\end{align*}
\]

This has the unique solution \(a = 2, b = 1, c = 0, d = 3\).

\[ f(t) = 2t^3 + t^2 + 3 \]

Problem 2 (10 points):

Let \(L\) be the line spanned by the vector \(\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}\). Write down the matrix representing the orthogonal projection onto the line \(L\) in \(\mathbb{R}^4\).

Answer.

\[
\begin{pmatrix}
1/4 & -1/4 & 1/4 & -1/4 \\
-1/4 & 1/4 & -1/4 & 1/4 \\
1/4 & -1/4 & 1/4 & -1/4 \\
-1/4 & 1/4 & -1/4 & 1/4
\end{pmatrix}
\]

Problem 3 (10 points):

Is the vector \(\vec{x} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}\) in the linear span of \(\vec{v} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}\) and \(\vec{w} = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}\)? If so, find real numbers \(a, b\) such that \(a \vec{v} + b \vec{w} = \vec{x}\).

Answer.

By reducing the matrix \(\begin{pmatrix} 2 & 4 & 3 \\ 4 & 3 & 1 \\ 0 & 5 & 5 \end{pmatrix}\) or by any other means, \(\frac{3}{5} = \frac{-1}{2} \begin{pmatrix} 2 \\ 4 \end{pmatrix} + 1 \begin{pmatrix} 4 \\ 3 \end{pmatrix}\).
Problem 4 (10 points):

In geometry, the centroid of a triangle is given by a simple formula. If the vertices of a triangle are at the points \((x_1, y_1), (x_2, y_2), (x_3, y_3)\), then the centroid of that triangle is at the point \(
\left( \frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)\).

We can think of this formula as defining a “centroid map” \(T: \mathbb{R}^6 \rightarrow \mathbb{R}^2\); let \(T \begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}\), where \((s, t)\) is the centroid of the triangle whose vertices \((u, v), (w, x), (y, z)\). This \(T\) is a linear transformation.

a) Write down the matrix that represents \(T\).

b) What is the rank of \(T\)?

**Answer.**

\(T\) is represented by \(\begin{bmatrix} 1/3 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 1/3 & 0 & 1/3 \end{bmatrix}\).

The rank is 2.

Problem 5 (10 points):

Alice and Bob are studying a two dimensional subspace of \(\mathbb{R}^{100}\) relative to bases \(A\) and \(B\). They are particularly interested in three vectors \(\vec{x}, \vec{y}, \vec{z}\).

In Alice’s coordinates, we have \([\vec{x}]_A = \begin{bmatrix} 8 \\ 4 \end{bmatrix}\), \([\vec{y}]_A = \begin{bmatrix} 3 \\ 5 \end{bmatrix}\), \([\vec{z}]_A = \begin{bmatrix} 1 \\ 11 \end{bmatrix}\).

In Bob’s coordinates, we have \([\vec{x}]_B = \begin{bmatrix} 4 \\ 0 \end{bmatrix}\), \([\vec{y}]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\).

How is \(\vec{z}\) written in \(B\)-coordinates?

**Answer.**

Note \(\begin{bmatrix} 1 \\ 11 \end{bmatrix} = 3\begin{bmatrix} 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix}\). So \(\vec{z} = 3\vec{y} - \vec{x}\), and that will be true in any coordinate system. \([\vec{z}]_B = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}\).

(This problem can alternately be done by computing a change-of-basis matrix.

Problem 6 (15 points):

Consider the matrix

\[
A = \begin{pmatrix}
1 & 1 & 0 & 3 & 1 \\
2 & 0 & 2 & 0 & 4 \\
1 & 0 & 1 & 0 & 7 \\
2 & 1 & 1 & 3 & 2
\end{pmatrix}
\]

and let \(T\) be the transformation represented by \(A\).

a) Write down a basis for the kernel of \(T\).

b) Write down a basis for the image of \(T\).
c) What is the rank (the dimension of the image) of $A$?

d) What is the nullity (the dimension of the kernel) of $A$?

**Answer.**

First, we row reduce the matrix to find $\text{rref } A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

A basis of $\ker T$ is $\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} -3 \\ -1 \\ 0 \\ 0 \end{pmatrix}$.

A basis of $\im T$ is $\begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 4 \\ 7 \\ 2 \end{pmatrix}$.

The rank is 3.

The nullity is 2.

**Problem 7 (15 points):**

Consider the following matrices.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

a) Compute $C = AB$.
b) Compute $A^{-1}$ or explain why $A$ is not invertible.
c) Compute $B^{-1}$ or explain why $B$ is not invertible.
d) Compute $C^{-1}$ or explain why $C$ is not invertible. (*Hint: there is more than one way to do this part.*)

**Answer.**

$$C = \begin{pmatrix} 4 & -1 & 3 \\ 1 & -1 & 2 \\ -1 & -1 & 1 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix}.$$  

$$C^{-1} = B^{-1}A^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ -3 & 7 & -5 \\ -2 & 5 & -3 \end{pmatrix}.$$  

**Problem 8 (20 points):**

Indicate whether each item is true or false. Each item is worth 2 points. No explanation necessary, no partial credit possible.

a) If $A$ and $B$ are invertible matrices and $AB = BA$, then also $A^{-1}B^{-1} = B^{-1}A^{-1}$.

**TRUE**
b) If $\vec{v}$ and $\vec{w}$ are in the kernel of a linear transformation $T$, then $\vec{v} + \vec{w}$ is also in the kernel of $T$.

**TRUE**

c) If two matrices have the same rref, then they have the same kernel.

**TRUE**

d) If $V$ and $W$ are 2-dimensional subspaces of $\mathbb{R}^3$ and $V \neq W$, then the set of vectors in both $V$ and $W$ is a 1-dimensional subspace of $\mathbb{R}^3$.

**TRUE**

e) If a linear transformation is invertible, then the inverse is linear.

**TRUE**

f) There exists a system of three equations in four variables what has a unique solution.

**FALSE**

g) There exists a system of three equations in four variables which has no solutions.

**TRUE**

h) The kernel of
\[
\begin{pmatrix}
2 & 1 & 0 & 2 \\
1 & 0 & 0 & 3 \\
7 & 3 & 0 & 4 \\
1 & 1 & 1 & 1
\end{pmatrix}
\]
is a subspace of $\mathbb{R}^4$.

**TRUE**

i) If $\vec{u}, \vec{v}, \vec{w}, \vec{x}$ are linearly independent vectors in $\mathbb{R}^n$, then $n \geq 4$.

**TRUE**

j) If $\{\vec{a}_1, \vec{a}_2, \vec{a}_3, ..., \vec{a}_k\}$ and $\{\vec{b}_1, \vec{b}_2, \vec{b}_3, ..., \vec{b}_\ell\}$ are bases of the same vector space, then $k = \ell$.

**TRUE**