This is an 80-minute exam, but you have the full 110 minute class period to complete it. There are a total of 100 points possible; the value of each problem is marked. Partial credit may be given for significant progress toward the solution to a problem. Except where otherwise indicated, you must show sufficient work to make it clear to me what you are doing and why.

**Problem 1: (10 points)**

Compute the determinant of following matrix by whatever method seems best to you.

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 2 & 3 & 3 \\
1 & 2 & 3 & 4
\end{pmatrix}
\]

**Problem 2: (10 points)**

Consider the subspace of \(\mathbb{R}^5\) spanned by \[\begin{pmatrix} 2 \\ 2 \\ 3 \\ 2 \\ 2 \end{pmatrix}\] and \[\begin{pmatrix} 2 \\ 4 \\ 0 \\ 4 \\ 2 \end{pmatrix}\]. Compute an orthonormal basis of this space.
Problem 3: (10 points)

Consider the three-dimensional subspace $V$ of $\mathbb{R}^4$ defined by $w + x + y + z = 0$, which has a basis $B = \left\{ \vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \right\}$.

(a) The vector $\vec{v} = \begin{bmatrix} -3 \\ 1 \\ -2 \\ 4 \end{bmatrix}$ is in the plane $V$. Give the $B$-coordinates of $\vec{v}$.

(b) The vector $\vec{w} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix}$ is NOT in the space $V$. Compute $\text{proj}_V \vec{w}$. 
Problem 4:  (10 points)

Consider the following linear system in three variables $x, y, z$.

\[
\begin{align*}
3x + y + 2z &= 1 \\
4x + 3y + \lambda z &= 0 \\
\lambda x + z &= 0
\end{align*}
\]

The coefficient $\lambda$ varies, so we want to solve for $x, y, z$ in terms of $\lambda$.

(a) This system has a unique solution, except for two values of $\lambda$. Which two values are these?

(b) Solve the system, assuming $\lambda$ is not one of the values from the previous part (some or all your answers may involve $\lambda$).
Problem 5:  (10 points)

Compute the least-squares solution to the following inconsistent system of equations. Please clearly indicate the normal equation.

\[
\begin{align*}
x + y &= 1 \\
2x + 8y &= -2 \\
x + 5y &= 3
\end{align*}
\]
Problem 6: (15 points)

Let $V$ be the plane in $\mathbb{R}^4$ spanned by $\vec{v}_1 = \begin{bmatrix} 2 \\ 3 \\ 6 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 7 \\ 4 \\ 12 \\ 6 \end{bmatrix}$. It turns out that the $QR$ decomposition of

$$
\begin{bmatrix}
2 & 7 \\
3 & 4 \\
6 & 12 \\
0 & 6
\end{bmatrix}
$$

is

$$
Q = \begin{bmatrix}
2/7 & 3/7 \\
3/7 & -2/7 \\
6/7 & 0 \\
0 & 6/7
\end{bmatrix},
R = \begin{bmatrix}
7 & 14 \\
0 & 7
\end{bmatrix}.
$$

(You don’t need to compute this for yourself.) Let $A$ be the basis $\vec{v}_1, \vec{v}_2$ and let $B$ be the basis given by the columns of $Q$.

(a) Consider the vector $\vec{w}$, which has $A$-coordinates $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Give the $B$-coordinates of $\vec{w}$.

(b) Compute the length of $\vec{w}$.

(c) Compute the volume of the parallelepiped generated by $\vec{v}_1$ and $\vec{v}_2$.  

Problem 7: (15 points)

Consider the map $T : U^{2 \times 2} \rightarrow U^{2 \times 2}$ defined by $T(M) = M \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} M$.

(a) Write the matrix representation of $T$, relative to the basis \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\} of $U^{2 \times 2}$.

(b) Write down a basis for the kernel of $T$.

(c) Write down a basis for the image of $T$.

(d) What is the rank of $T$?

(e) What is the nullity of $T$?

(f) What is the determinant of $T$?
Problem 8: (20 points)

Each item is worth 2 points. No explanation necessary, no partial credit possible.

(a) If $A$ and $B$ are similar matrices, then $A$ and $B$ have the same rank.

True  False

(b) $P_4$, the space of all polynomial of degree at most 4, has a basis consisting entirely of fourth-degree polynomials.

True  False

(c) A linear map $T : V \to W$ is an isomorphism if and only if it is represented by an invertible matrix (relative to some choice of coordinates).

True  False

(d) If $T : V \to V$ is a linear map from a linear space to itself, $\ker T$ and $\operatorname{im} T$ have nothing in common except the zero element of $V$.

True  False

(e) If $V$ is a 3-dimensional subspace of $\mathbb{R}^5$, $V^\perp$ is a plane.

True  False

(f) If $T : \mathbb{R}^n \to \mathbb{R}^n$ has the property that the angle between $T(\vec{v})$ and $T(\vec{w})$ is always the same as the angle between $\vec{v}$ and $\vec{w}$, then $T$ is an orthogonal transformation.

True  False

(g) If a matrix has orthonormal rows, it also has orthonormal columns.

True  False

(h) If $A, B$ are $n \times n$ matrices, $\det A = 7$ and $\det B = 3$, then $\det(AB) = 21$.

True  False

(i) If $A$ is invertible, $A^T$ is also invertible.

True  False

(j) If $A, B$ are $n \times n$ matrices, $\det A = 7$ and $\det B = 3$, then $\det(A+B) = 10$.

True  False