AN IMPERFECT RING WITH A TRIVIAL COTANGENT COMPLEX

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Fix a perfect field \( k \) of characteristic \( p \). Recall the following well-known fact:

**Proposition 0.1.** If \( A \) is a perfect \( k \)-algebra, then \( L_{A/k} \simeq L_A \simeq 0 \).

**Proof.** The Frobenius \( F : A \to A \) induces the 0 map \( F_* : L_A \to L_A \) (which is true for any \( F_p \)-algebra \( A \)). On the other hand, since \( A \) is perfect, \( F \) is an isomorphism, so \( F_* \) is also an isomorphism, and thus \( L_A \simeq 0 \). Finally, since \( k \) is perfect, \( L_{A/k} \simeq L_A \) as \( L_k = 0 \). \( \square \)

The goal of this note is to record a counterexample to the converse statement; the idea of forcing variables to become products, rather than powers, used below was suggested to me by Gabber, and I grateful to him for this suggestion.

**Proposition 0.2 (Gabber).** There exists a non-reduced \( k \)-algebra \( A \) such that \( L_{A/k} \simeq L_A \simeq 0 \).

**Proof.** For \( i \geq 0 \), let \( B_i = k[x_{i,1}, x_{i,2}, \ldots, x_{i,2^i}] \) be the polynomial algebra on the displayed \( 2^i \) generators. Write \( I_i \subset B_i \) for the ideal spanned by the variables, and set \( A_i = B_i,_{\text{perf}}/J_i \), where \( B_i,_{\text{perf}} \) is the perfection of \( B_i \) (i.e., the direct limit along Frobenius), and \( J_i = I_i \cdot B_i,_{\text{perf}} \). Then \( L_{B_i,_{\text{perf}}} = 0 \).

As \( J_i \) is defined by a regular sequence, it is standard to see that \( L_{A_i} \simeq J_i/J_i^2[1] \) is a free module on \( 2^i \) generators, placed in homological degree 1. Now define maps \( A_i \to A_{i+1} \) given by

\[
x_{i,j}^p \mapsto \left( x_{i+1,2j} \cdot x_{i+1,2j+1} \right)^{p^i}.
\]

In other words, each variable in \( A_i \) becomes a product of two variables in \( A_{i+1} \). Set \( A = \text{colim} A_i \). Then we claim that \( L_A = 0 \), and yet \( A \) is non-reduced. To see \( L_A = 0 \), note that

\[
L_A \simeq \text{colim}_i L_{A_i} \simeq \text{colim}_i J_i/J_i^2[1],
\]

as the formation of the cotangent complex commutes with filtered colimits. Now it is enough to observe that the natural map

\[
J_i/J_i^2 \to J_{i+1}/J_{i+1}^2
\]

is the 0 map, since each variable in \( J_i \) becomes a product of two variables in \( J_{i+1} \). To see that \( A \) is not perfect, set \( \alpha := x_{0,1}^p \in A_0 \). Then \( \alpha^p = 0 \) (since \( x_{0,1} \in J_0 \)). On the other hand, the image of \( \alpha \) in \( A_i \) is given by

\[
\prod_{j=1}^{2^i} x_{i,j}^{p^j},
\]

which is non-zero (as it does not lie in \( J_i \)). Thus, \( \alpha \) gives a nilpotent non-zero element in \( A \), so \( A \) is not reduced. \( \square \)

We end by raising a question about the characteristic 0 analog:

**Question 0.3.** Let \( E \) be a field of characteristic 0. Does there exist an \( E \)-algebra \( A \) such that \( L_{A/E} \simeq 0 \), yet \( A \) is not ind-étale over \( E \)?