DERIVED DIRECT SUMMANDS

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Abstract

We study certain questions in positive and mixed characteristic algebraic geometry related to the direct summand conjecture.

Our main result in positive characteristic is that the direct summand condition (see Condition 1.0.2) coincides with a derived enhancement (see Condition 1.0.1); one can view this result as asserting that the singularities satisfying the direct summand condition define a good positive characteristic analogue of the rational singularities of characteristic 0. Using this theorem, we are able to prove a number of results which, roughly speaking, assert that vanishing theorems familiar from complex geometry have positive characteristic analogues provided we ask for vanishing "up to passage to finite covers." Moreover, our results are sharp in the sense that we give examples illustrating the necessity of our hypotheses.

We prove two theorems in mixed characteristic. The first is an analogue of the positive characteristic theorem alluded to above, except that "vanishing" is replaced by "divisibility by p." Our proof of this theorem also provides a new proof of the pure positive characteristic result mentioned above. The second is that the direct summand conjecture holds in cases where the ramification is supported on a simple normal crossings divisor. To the best of our knowledge, this is the first family of examples where the direct summand conjecture is proven without putting any restrictions on the dimension; our proof uses methods from p-adic Hodge theory.

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To my parents.

Contents

Chapter 1

Introduction

The study of singular varieties, as limits of smooth varieties in moduli theory or as the objects of intrinsic interest in higher dimensional geometry, is inevitable in modern algebraic geometry. The results of this thesis pertain to certain basic features of a particularly nice class of singularities known as "rational singularities".

Definition. *A scheme S has* rational singularities *if there is a resolution $f : X \to S$ with $\mathcal{O}_S \to \mathrm{R}f_*\mathcal{O}_X$ an isomorphism*[1].

Standard examples of rational singularities are quotient singularities and cones on Fano hypersurfaces. The flawed yet useful intuition informing the general definition is that such varieties admit resolutions with rational exceptional fibres and hence satisfy some local positivity. The resulting class of singularities enjoys many pleasant features: independence of choice of resolution, closure under deformations and quotients, and good Hodge-theoretic properties. The dependence on resolutions, however, renders this definition inapplicable to positive or mixed characteristic geometry. Recently, Sándor Kovács discovered an intrinsic characterisation of rational singularities using:

Condition 1.0.1. For any proper surjective morphism $f : X \to S$, the natural map $\mathcal{O}_S \to \mathrm{R}f_*\mathcal{O}_X$ is split in $\mathrm{D}(\mathrm{Coh}(S))$.

Kovács showed in [Kov00] that a complex variety S satisfies Condition 1.0.1 exactly when it has rational singularities. This result is useful as Condition 1.0.1 is often easy to verify, e.g., canonical singularities satisfy Condition 1.0.1 thanks to the Grauert-Riemenschneider vanishing theorem and are thus rational. My own work attempts to understand Condition 1.0.1 in positive and mixed characteristic, by relating it to an analogous condition involving only finite maps:

Condition 1.0.2. For any finite surjective morphism $f : X \to S$, the natural map $\mathcal{O}_S \to f_*\mathcal{O}_X$ is split in $\mathrm{Coh}(S)$.

As finite maps are proper, any scheme satisfying Condition 1.0.1 also satisfies Condition 1.0.2; the converse fails in characteristic 0 as all normal schemes satisfy Condition 1.0.2 by a trace argument, while not all normal schemes have rational singularities. Away from characteristic 0, however, we discover a remarkably different picture. We summarise our results in the next section.

[1] The tradtional definition (see [KKMSD73, Chapter I, §3, page 50]) of rational singularities also requires that $\mathrm{R}^i f_*\omega_X = 0$ for $i > 0$, where ω_X is the canonical line bundle on X. This last condition is automatic in characteristic 0 by Grauert-Riemenschneider vanishing (see [Laz04a, Theorem 4.3.9]), but not true in general. Throughout this thesis, all references to rational singularities will be made for schemes in characteristic 0, and consequently, this distinction will not concern us.

Summary of results

This thesis is organised as follows. The purpose of Chapter 2 is to setup some notation, and review some relevant notions in duality theory. Chapters 3 and 4 are preliminary in nature, and contain mainly examples. Chapters 5 through 7 study Condition 1.0.2 in positive characterstic p, though the results in Chapter 6 are characteristic independent. The last two chapters, Chapters 8 and 9, study mixed characteristic $(0, p)$ analogues of questions studied earlier. A more detailed description of the contents and results of each chapter (except Chapter 2) follows.

We begin with a preliminary study of Conditions 1.0.2 and 1.0.1 in Chapters 3 and 4. In each case, we first understand the conditions in characteristic 0 in terms of classical notions in algebraic geometry, and then give some examples supporting the intuition that these conditions measure positivity in positive characteristic. The emphasis in these chapters is on examples illustrating representative behaviour rather than general theorems.

Chapter 5 marks our first serious foray into understanding these conditions in positive characteristic. The main theorem of this chapter (see Theorem 5.0.2) is:

Theorem 1.0.3. *Conditions 1.0.1 and 1.0.2 are equivalent for noetherian* $\mathbf{F}_p$*-schemes.*

The proof of Theorem 1.0.3 is inspired by certain well-known results in commutative algebra originally due to Hochster and Huneke [HH92], and refined by [HL07]. While geometrising these results, Karen Smith arrived at certain questions that we are able to answer. Our answer is summarised below; we refer the reader to §5.5 for a detailed discussion.

Theorem 1.0.4. *Let* X *be a proper variety over a field* k *of positive characteristic, and let* $\mathcal{L} \in \mathrm{Pic}(X)$*. If* $\mathcal{L}$ *is semiample, then* $H^i(X, \mathcal{L})$ *can be killed by finite covers for* $i > 0$*; this conclusion fails if* $\mathcal{L}$ *is only assumed to be nef. If* $\mathcal{L}$ *is semiample and big, then* $H^i(X, \mathcal{L}^{-1})$ *can be killed by finite covers for* $i < \dim(X)$*; this conclusion fails if either semiampleness or bigness is not assumed (in particular, it fails in the nef and big case).*

The main technique underpinning the results of Chapter 5 is a method for constructing finite covers that annihilate coherent cohomology under suitable finiteness assumptions (see Proposition 5.2.2). This technique, originally due to Hochster and Huneke and dubbed the "Equational Lemma," has undergone a series of reformulations starting with [HH92], through [Smi94], and ending with [HL07]. Despite these revisions, it retains a certain *ad hoc* character, essentially because it involves manipulations with cocycles in *coherent* cohomology using Frobenius. Chapter 6 provides a new proof of this result using general results on group schemes. Our approach has the advantage of providing perhaps a more conceptual understanding of the cocycle manipulations by embedding them in a larger topological context. Our main theorem is the following result on the fppf cohomology of group schemes (see Theorems 6.0.1 and 6.0.2); the coherent cohomology claims are deduced using subgroup schemes of $\mathbf{G}_a$ defined by additive polynomials acting as endomorphisms of $\mathbf{G}_a$.

Theorem 1.0.5. *Let* S *be a noetherian excellent scheme. Given a finite flat group scheme* $G \to S$*, cohomology classes in* $H^n(S, G)$ *can be killed by finite covers for* $n > 0$*. Given an abelian scheme* $A \to S$*, cohomology classes in* $H^n(S, A)$ *can be killed by proper covers for* $n > 0$*.*

In Chapter 7, we relate Condition 1.0.2 to preexisting notions notions of rational singularities in positive characteristic. Two of the most significant ones are pseudorationality due to Lipman and Tessier [LT81], and F-rationality originally due to Fedder and Watanabe [FW89], and further studied by Karen Smith. Our main theorem (see Theorems 7.1.4 and 7.2.5) indicates that there is a close connection between all three:

Theorem 1.0.6. *Let R be an excellent $\mathbf{F}_p$-algebra. If R satisfies Condition 1.0.2, then R is F-rational and pseudorational. If R is Gorenstein and F-rational, then R satisfies Condition 1.0.2.*

In Chapters 8 and 9, we move from positive characteristic p to mixed characteristic $(0, p)$. Chapter 8 studies the mixed characteristic version of the questions studied in Chapter 5. We are unable to prove an exact analogue of Theorem 1.0.3 in mixed characteristic as we lack Frobenius. Nevertheless, we can prove an analogue of the cohomology annihilation theorem alluded to earlier.

Theorem 1.0.7. *Given a proper morphism $f : X \to \mathrm{Spec}(A)$ with A excellent, there exists proper cover $\pi : Y \to X$ such that $\pi^*(H^i(X, \mathcal{O}_X)) \subset p(H^i(Y, \mathcal{O}_Y))$ for all $i > 0$.*

Our proof of Theorem 1.0.7 is geometric, relying ultimately on the study of semistable curve fibrations. Using Theorem 1.0.7, we are able to reprove Theorem 1.0.3 in yet another way in §8.2.

In Chapter 9, we study the direct summand conjecture. This conjecture asserts that regular rings satisfy Condition 1.0.2, and formed our original motivation for studying Conditions 1.0.2 and 1.0.1. Under the analogy with rational singularities, this conjecture can be viewed as asking if smooth schemes have rational singularities. The conjecture is easy in characteristic 0 (see Proposition 3.1.5), known in characteristic p by work of Hochster (see Theorem 3.2.1), and known in mixed characteristic in dimension ≤ 3 by work of Heitmann [Hei02]. In Chapter 9, we study this conjecture in mixed characteristic using techniques and results from Faltings' theory of almost étale extensions (see [Fal02]), originally discovered in p-adic Hodge theory. Our main result is that the direct summand conjecture holds in arbitrary dimensions when the ramification is supported on a normal crossings divisor (see Theorem 9.0.1):

Theorem 1.0.8. *Let R be a p-adic ring that is smooth over a finite extension of $\mathbf{Z}_p$, and let $f : R \to S$ be the normalisation of R in a finite extension of its fraction field. Assume f is étale away from a divisor with normal crossings. Then f is split as an R-module map.*

A word on examples: we have tried hard to find many examples of most conditions, definitions, and theorems appearing in this thesis. The more interesting amongst these are: Example 4.2.4, Corollary 4.2.8, §5.5.2, Proposition 5.6.5, §6.5, Example 7.3.3, and Example 9.1.8.

Chapter 2

Conventions

We list a few conventions followed throughout this thesis. Along the way, we also summarise relevant background material in Grothendieck duality theory, if only to setup notation.

1. Finiteness conditions: All schemes occuring in Chapters 3 through 8 are assumed to be noetherian.

2. Quasicoherent sheaves: We let $\mathrm{QCoh}(X)$ denote the category of quasicoherent sheaves on a scheme X. When X is affine, we identify objects in $\mathrm{QCoh}(X)$ with modules over $\Gamma(X, \mathcal{O}_X)$. We adopt similar conventions for the category $\mathrm{Coh}(X)$ of coherent sheaves.

3. Derived categories: Given a noetherian scheme X, we use $\mathrm{D}(\mathrm{Coh}(X))$ to denote the derived category of $\mathrm{Coh}(X)$. For integers $m, n \in \mathbf{Z}$ with $m \leq n$, we use $\mathrm{D}^{[m,n]}(\mathrm{Coh}(X)$ to denote the full subcategory of $\mathrm{D}(\mathrm{Coh}(X))$ spanned by complexes K with $\mathcal{H}^i(K) = 0$ for $i < m$ or $i > n$. We make similar definitions for $D^b(\mathrm{Coh}(X))$ (all complexes with bounded cohomology), $D^{\leq n}(\mathrm{Coh}(X))$, etc. As a reference for triangulated categories and t-structures, we suggest [BBD82, §1].

4. Dualising complexes: A scheme X possessing a dualising complex $\omega_X^\bullet$ in the sense of [Har66, Chapter V, §2] always has the following normalisation: the dualising sheaf ω_X as defined in [Har77, Chapter III, §7] occurs as $\mathcal{H}^{-d}(\omega_X^\bullet)$. Thus, we have $\omega_X^\bullet \in \mathrm{D}^{[-d,0]}(\mathrm{Coh}(X))$, where $d = \dim(X)$.

5. Local duality: Let $(S, s) = (\mathrm{Spec}(R), \mathfrak{m})$ be a noetherian local scheme possessing a dualising complex $\omega_R^\bullet$. The local duality functor D (sometimes referred to as Matlis duality) is defined by $\mathrm{Hom}_S(-, E)$ where E is an injective hull of the residue field $R/\mathfrak{m}$. Once a dualising complex $\omega_R^\bullet$ has been fixed, the hull E may be identified with $\mathrm{R}\Gamma_\mathfrak{m}(\omega_R^\bullet)$. This functor is exact, contravariant, length preserving, and transforms ind-artinian $\mathcal{O}_S$-modules to pro-artinian $\mathcal{O}_S$-modules. The duality theorem asserts that applying $\mathrm{R}\Gamma_\mathfrak{m}(-)$ induces a functorial equivalence

$$\mathrm{R}\widehat{\mathcal{H}om(\mathcal{F}, \omega_R^\bullet)} \simeq D(\mathrm{R}\Gamma_\mathfrak{m}(\mathcal{F}))$$

   for every $\mathcal{F} \in \mathrm{D}^b(\mathrm{Coh}(S))$. For more details, we refer the reader to [Har66, Chapter V, §6].

6. Localising local duality: Let $(S, s) = (\mathrm{Spec}(R), \mathfrak{m})$ be a noetherian local scheme possessing a dualising complex $\omega_R^\bullet$, and let $\mathfrak{p} \subset R$ be a prime ideal. Our chosen normalisation for dualising complexes implies that $(\omega_R^\bullet)_\mathfrak{p} \simeq \omega_{R_\mathfrak{p}}^\bullet[c_\mathfrak{p}]$ where $c_\mathfrak{p} = \dim(R) - \dim(R_\mathfrak{p})$ is the

codimension of $\mathfrak{p}$. Hence, for any $\mathcal{F} \in \mathrm{D}^b(\mathrm{Coh}(S))$ and any integer i, we have

$$\mathcal{H}^{-d+i}(\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{F}, \omega_R^\bullet))_\mathfrak{p} \simeq \mathcal{H}^{-d_{R_\mathfrak{p}}+i}(\mathrm{R}\mathcal{H}\mathrm{om}(\mathcal{F}_\mathfrak{p}, \omega_{R_\mathfrak{p}}^\bullet))$$

where $d = \dim(R)$, and $d_{R_\mathfrak{p}} = \dim(R_\mathfrak{p})$. Similarly, we also have

$$D(H_\mathfrak{m}^{d-i}(\mathcal{F}))_\mathfrak{p} \simeq D(H_\mathfrak{p}^{d_{R_\mathfrak{p}}-i}(\mathcal{F}_\mathfrak{p})).$$

One can summarise the above by saying that "codegree" of the local cohomology localises well. For more details, we refer the reader to [Har66, Chapter V, §6].

7. Grothendieck duality: A proper map $f : X \to S$ of schemes possessing dualising complexes induces a trace map $\mathrm{Tr}_f : \mathrm{R}f_*\omega_X^\bullet \to \omega_S^\bullet$ which, in turn, gives rise to a functorial equivalence

$$\mathrm{R}f_*\mathrm{R}\mathcal{H}\mathrm{om}_X(\mathcal{F}, \omega_X^\bullet) \simeq \mathrm{R}\mathcal{H}\mathrm{om}_S(\mathrm{R}f_*\mathcal{F}, \omega_S^\bullet)$$

for every $\mathcal{F} \in \mathrm{D}^b(\mathrm{Coh}(X))$ (see [Har66, Chapter VII, Theorem 3.3]).

8. Serre duality: This duality is the special case of Grothendieck dualty when $S = \mathrm{Spec}(k)$ is the spectrum of a field, and that X is proper, Cohen-Macaulay, and of equidimension n. Given $\mathcal{F} \in \mathrm{D}^b(\mathrm{Coh}(X))$, there is a functorial isomorphism

$$H^i(X, \mathcal{F})^\vee \simeq \mathrm{Ext}_X^{-i}(\mathcal{F}, \omega_X^\bullet) \simeq \mathrm{Ext}_X^{n-i}(\mathcal{F}, \omega_X).$$

The functoriality implies the following: given a map $\mathcal{F} \to \mathcal{G}$ in $\mathrm{D}^b(\mathrm{Coh}(X))$, the induced maps

$$H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G})$$

and

$$\mathrm{Ext}_X^{n-i}(\mathcal{G}, \omega_X) \to \mathrm{Ext}_X^{n-i}(\mathcal{F}, \omega_X)$$

are dual to each other as maps of finite dimensional vector spaces for each integer i.

Chapter 3

The direct summand condition

In this chapter, we discuss the direct summand condition. Recall that this condition says:

Condition (1.0.2)**.** *Given a finite surjective morphism $f : X \to S$, the morphism $\mathcal{O}_S \to f_*\mathcal{O}_X$ admits a section in the category* $\mathrm{Coh}(S)$ *of coherent sheaves on* S.

The plan for this chapter is as follows: in §3.1 we discuss the equivalence of Condition 1.0.2 with normality in characteristic 0, and in §3.2 we discuss the analogy between Condition 1.0.2 in positive characteristic p and the theory of rational singularities by virture of examples.

3.1 The situation in characteristic 0

Condition 1.0.2 in characteristic 0 is classical and simple: it is equivalent to normality. To prove this claim, we first verify that irreducibility is necessary.

Lemma 3.1.1. *Let S be a connected noetherian scheme with irreducible components $S_1, \ldots, S_k$ given their reduced structure. Assume $k \geq 2$. Then the map $i : \mathcal{O}_S \to \prod_i \mathcal{O}_{S_i}$ has no splitting in* $\mathrm{Coh}(S)$.

Proof. Assume to the contrary that there was a section. Then the cokernel $\mathcal{Q} = \mathrm{coker}(i)$ can be viewed as a subsheaf of $\prod_i \mathcal{O}_{S_i}$. This sheaf is supported exactly at those points of S through which more than one of the S_i's pass. As each S_i is a maximal irreducible closed subset, the preceding implies that the image of the natural map $\mathcal{Q} \to \mathcal{O}_{S_i}$ is supported on a proper subset of S_i. On the other hand, each S_i is an integral scheme and, consequently, its structure sheaf $\mathcal{O}_{S_i}$ does not contain a non-zero submodule supported along a proper closed subset. Thus, the image of $\mathcal{Q}$ is 0 in each $\mathcal{O}_{S_i}$. This implies that $\mathcal{Q} = 0$ and, consequently, that $\mathcal{O}_S \simeq \prod_i \mathcal{O}_{S_i}$. However, this is clearly false because the fibres of the two sheaves differ at a point contained in more than one of the S_i's, and such a point has to exist by connectedness. Thus, we reach a contradiction. □

Next, we show that we can always refine the source without alterting the truth of Condition 1.0.2. The proof is trivial, but we record it for completeness.

Lemma 3.1.2. *Let $f : X \to S$, and $\pi : Y \to X$ be morphism of schemes. If $\mathcal{O}_S \to f_*\pi_*\mathcal{O}_Y$ has a section, so does $\mathcal{O}_S \to f_*\mathcal{O}_X$.*

Proof. We have a factorisation $\mathcal{O}_S \to f_*\mathcal{O}_X \to f_*\pi_*\mathcal{O}_Y$ in $\mathrm{Coh}(S)$. Hence, if the composite has a section, so does the first map. □

We next show that Condition 1.0.2 passes to summands. The proof is again trivial, but the observation turns out to be useful, especially in the sequel.

Lemma 3.1.3. *Let $\pi : U \to X$ be a morphism such that $\mathcal{O}_X \to \pi_*\mathcal{O}_U$ has a section. Then X satisfies Condition 1.0.2 if U does so.*

Proof. Let $g : Z \to X$ be a finite surjective morphism. Consider the diagram

$$\begin{array}{ccc} Z_U = Z \times_X U & \xrightarrow{a} & U \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{g} & X \end{array}$$

The maps a and b are defined by the diagram. This square gives rise to the following commutative square in $\mathrm{Coh}(X)$:

$$\begin{array}{ccc} (g \circ b)_*\mathcal{O}_{Z_U} \simeq (\pi \circ a)_*\mathcal{O}_{Z_U} & \longleftarrow & \pi_*\mathcal{O}_U \\ \uparrow & & \uparrow \\ g_*\mathcal{O}_Z & \longleftarrow & \mathcal{O}_X \end{array}$$

The arrows on the right and the top are split by assumption. It follows formally that the same is true for the bottom arrow, as desired. □

We now show that Condition 1.0.2 implies its ind-finite version; the additional flexibility of provided by splitting off ind-finite morphisms is useful in avoiding excellence assumptions.

Proposition 3.1.4. *Let S be a noetherian scheme satisfying Condition 1.0.2, and let $f : X \to S$ be a surjective morphism of schemes such that $\mathcal{O}_S \to f_*\mathcal{O}_X$ is an ind-finite morphism of quasicoherent $\mathcal{O}_S$-algebras. Then $\mathcal{O}_S \to f_*\mathcal{O}_X$ is split in $\mathrm{Coh}(S)$.*

Proof. By Lemma 3.1.1, we may assume that S is integral. Hence, the surjectivity of f gives an exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_X \to \mathcal{Q} \to 0$$

with $\mathcal{Q}$ defined as the cokernel. This sequence defines a class $\mathrm{ob}(f) \in \mathrm{Ext}^1_S(\mathcal{Q}, \mathcal{O}_S)$, and our goal is to show this class vanishes. The assumption implies that $X = \lim X_i$, where $f_i : X_i \to S$ is a finite surjective morphism, and the indexing set is cofiltered. Hence, we can write $f_*\mathcal{O}_X$ as a filtered colimit of coherent $\mathcal{O}_S$-algebras:

$$f_*\mathcal{O}_X = \mathrm{colim}\, f_{i*}\mathcal{O}_{X_i}.$$

Each index i gives an exact sequence

$$0 \to \mathcal{O}_S \to f_*\mathcal{O}_{X_i} \to \mathcal{Q}_i \to 0$$

where $\mathcal{Q}_i$ is defined as the cokernel. The formula for $f_*\mathcal{O}_X$ and the right exactness of colimits gives the formula

$$\mathcal{Q} = \mathrm{colim}_i\, \mathcal{Q}_i.$$

Since filtered colimits are exact, this formula translates to

$$\mathrm{Ext}^1_S(\mathcal{Q}, \mathcal{O}_S) \simeq \lim_i \mathrm{Ext}^1_S(\mathcal{Q}_i, \mathcal{O}_S)$$

with the limit on the right cofiltered. The functoriality of the construction implies that $\mathrm{ob}(f)$ maps to 0 in each $\mathrm{Ext}^1_S(\mathcal{Q}_i, \mathcal{O}_S)$; the preceding formula then implies that $\mathrm{ob}(f) = 0$, as desired. □

We can now prove the promised result.

Proposition 3.1.5. *A noetherian* $\mathbf{Q}$*-scheme* S *satisfies the direct summand condition if and only if* S *is normal. In particular, the property that "*S *satisfies Condition 1.0.2" is local on* S *under the preceding assumptions.*

Proof. As both conditions are stable under taking connected components, we may assume that S is connected for either implication. Now assume that S is a noetherian $\mathbf{Q}$-scheme satisfying Condition 1.0.2. We will verify that S is normal. By Lemma 3.1.1, we may assume that S is integral. Let $f : S' \to S$ be the normalisation of S in its fraction field. By Proposition 3.1.4, the map $\mathcal{O}_S \to f_*\mathcal{O}_{S'}$ has a section. It now suffices to verify the following ring-theoretic statement: if R is an integral domain with normalisation R', then $R \to R'$ is a direct summand only if R is normal. If $R \to R'$ is a direct summand, then the complementary submodule to R in R' is a torsion-free R-module whose generic rank is 0. Such modules are forced to be trivial, and the claim follows.

For the converse implication, we need to show that if $f : X \to S$ is a finite surjective morphism, and S is normal and connected, then $f^* : \mathcal{O}_S \to f_*\mathcal{O}_X$ has a section in $\mathrm{Coh}(S)$. After replacing X with the normalisation of a dominating irreduicible component thanks to Lemma 3.1.2, we may assume that X is also normal and connected. Let d denote the degree of the map induced by f at the level of function fields. Then the map $\frac{1}{d}\mathrm{Tr}_{X/S}$ provides a canonical splitting for the map $f^* : \mathcal{O}_S \to f_*\mathcal{O}_X$ (here we use that the trace map on function fields preserves integrality). $\square$

Remark 3.1.6. In the proof of Proposition 3.1.5, the characteristic 0 assumption was only used for the converse implication while dividing by the trace map by the degree. Thus, any noetherian scheme satisfying Condition 1.0.2 is automatically normal.

3.2 The situation in characteristic p

Condition 1.0.2 in positive characteristic p is subtler than its characteristic 0 avatar: it measures some kind of positivity (both local and global) on the variety. In this section, we summarise some known examples (and non-examples) of schemes satisfying this condition. The intuition informing most of these results is that rings satisfying Condition 1.0.2 are analogous to rational singularities in characteristic 0, an intuition that will be justified to some extent in Chapters 5 and 7.

3.2.1 The case of regular schemes

The first result we discuss is that *regular affine* $\mathbf{F}_p$-schemes satisfy Condition 1.0.2. Under the afore mentioned analogy with the theory of rational singularities, this result may be understood as analogous to the fact that smooth varieties have rational singularities. We provide two proofs of this fact: the first is due to Mel Hochster (see [Hoc73]), and the second was discovered by the author. The latter is certainly obvious to the experts as it is essentially a translation of Hochster's proof into local cohomology; we decided to include it as it introduces ideas that will be useful in the sequel.

Proposition 3.2.1 (Hochster)**.** *Let* R *be an regular* $\mathbf{F}_p$*-algebra, and let* $f : R \to S$ *be finite extension, i.e.,* S *is a domain,* $\mathrm{Frac}(R) \to \mathrm{Frac}(S)$ *is a finite field extension, and* S *is finite over* R*. Then* f *admits an* R*-linear splitting* $\pi : S \to R$*.*

Hochster's proof. Let $\mathcal{Q}$ be the quotient S/R. Then the existence of an R-linear splitting of f amounts to the triviality of the extension class in $\mathrm{Ext}^1_R(\mathcal{Q}, R)$ determined by

$$0 \to R \to S \to \mathcal{Q} \to 0$$

As Ext groups localise well for the fpqc topology, we immediate reduce to the case that R is a complete regular local ring with maximal ideal $\mathfrak{m}$. Let $x_1, \dots, x_n$ be a regular sequence generating $\mathfrak{m}$, and set $\mathfrak{m}_k = (x_1^k, \dots, x_n^k)$ for any integer $k \geq 1$. With this notation, we claim that f is R-split if and only if $\mathfrak{m}_k S \cap R = \mathfrak{m}_k$. The forward implication is clear. To verify the converse implication, note that $\mathfrak{m}_k T \cap T = \mathfrak{m}_k$ implies that $f_k : R/\mathfrak{m}_k \to T/\mathfrak{m}_k T$ is injective. As the ring $R/\mathfrak{m}_k$ is Gorenstein, it is injective as a module over itself. Hence, the map f_k being injective implies that f_k is split. As $\{\mathfrak{m}_k\}_k$ forms a basis for the $\mathfrak{m}$-adic topology on R, we see that $f : R \to S$ is also split.

Given the preceding claim, it suffices to verify that $\mathfrak{m}_{k+1} S \cap R = \mathfrak{m}_{k+1}$ for each k. We may now choose an R-linear map $\phi : S \to R$ such that $\phi(1) \neq 0$ (choose one at the level of function fields and scale until it is integral). As $R/\mathfrak{m}_{k+1}$ is Gorenstein, the socle of $\mathfrak{m}/\mathfrak{m}_{k+1}$ is generated by the element $\prod_i x_i^k$. Thus, verifying $\mathfrak{m}_{k+1} S \cap R = \mathfrak{m}_{k+1}$ amounts to verifying that $\prod_i x_i^k \notin \mathfrak{m}_{k+1} T$. Assume towards contradiction that

$$\prod_i x_i^k = \sum_i s_i x_i^{k+1}$$

for some $s_i \in S$. The basic idea is that the failure of ϕ to be a section of i is bounded by $\phi(1)$, while applying Frobenius sufficiently many times to the preceding equation makes this failure unbounded. In more detail, applying ϕ to the preceding equation gives $\prod_i x_i^k \phi(1) \in \mathfrak{m}_{k+1}$. By the regularity of R, this means that $\phi(1) \in \mathfrak{m}$. On the other hand, running this argument again after applying Frobenius to the preceding equation tells us that $\phi(1) \in \mathfrak{m}_p$. Continuing this way, we see that $\phi(1) \in \mathfrak{m}_{p^k}$ for all k, which contradicts Krull's theorem asserting that $\cap_k \mathfrak{m}_{p^k} = (0)$. □

Second proof. We first explain the idea informally. Using the fact that R is Gorenstein, an elementary duality argument will reduce us to showing that $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(S)$ is injective. The kernel of this map is a Frobenius stable proper submodule of $H^d_{\mathfrak{m}}(R)$ of finite length by an inductive argument due to Grothendieck (see [Gro68a, Exposé VIII, Théorème 2.1]). The regularity of R is regular will then imply that this is impossible for length reasons.

Now for the details. After localising as in Hochster's proof, we may assume that $(R, \mathfrak{m})$ is a complete regular local $\mathbf{F}_p$-algebra of dimension d. By the Cohen structure theorem (see [Mat70, §28, Theorem 28.J and Corollary 2]), we know that $R \simeq k[\![x_1, \cdots, x_n]\!]$. By Lemma 3.1.3 and the fact that field extensions $k \to L$ split as k-modules, we may pass to the algebraic closure of the coefficient field to assume that k is algebraically closed. In particular, the Frobenius map $F : R \to R$ is finite. Given a finite extension $f : R \to R$, we need to show that the natural map $\mathrm{ev}_f : \mathrm{Hom}(S, R) \to \mathrm{Hom}(R, R)$ is surjective. By induction, we may assume that the cokernel $\mathcal{Q}$ is supported only at the closed point $\{\mathfrak{m}\} \subset \mathrm{Spec}(R)$. In particular, the cokernel $\mathcal{Q}$ has finite length. Now the fact that R is Gorenstein implies that $\omega_R \simeq R$. Thus, the map being considered is simply the trace map $\mathrm{Hom}(S, \omega_R) \to \omega_R$, which is dual to the canonical pullback map $H^d_{\mathfrak{m}}(f) : H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(S)$. As local duality interchanges kernels and cokernels while preserving lengths, it follows that the kernel $\ker(H^d_{\mathfrak{m}}(f))$ has the same length as $\mathcal{Q}$. This kernel is also Frobenius-stable by construction. On the other hand, since R is regular, the Frobenius map $F : R \to R$ is also flat in addition to being finite. The flat base change isomorphism $\mathrm{R}\Gamma_{\mathfrak{m}}(R) \otimes_{R,F} R \simeq \mathrm{R}\Gamma_{\mathfrak{m}}(R)$ (see [BS98, §4.3.2]) then shows that $H^d_{\mathfrak{m}}(R)$ has a natural F-crystal structure. It then follows that $\ker(H^d_{\mathfrak{m}}(f)) \subset H^d_{\mathfrak{m}}(R)$ is a finite length Frobenius-stable submodule of the F-crystal $H^d_{\mathfrak{m}}(R)$. By explicit calculation (see Lemma 3.2.2 below), however, we know that $H^d_{\mathfrak{m}}(R)$ has no finite length Frobenius stable submodules except the trivial one. Hence, we see that $\ker(H^d_{\mathfrak{m}}(f)) = 0$ and, therefore, that $\mathcal{Q} = 0$ as desired. □

Here is the elementary calculation that was needed in our proof of Proposition 3.2.1.

Lemma 3.2.2. *Let X be an regular $\mathbf{F}_p$-scheme of dimension d, and let $(\mathcal{E}, \phi)$ by a unit F-crystal on R, i.e., $\mathcal{E}$ is a quasi-coherent sheaf, and $\phi : F^*\mathcal{E} \simeq \mathcal{E}$ is an isomorphism, where F is the absolute Frobenius on X. If $\mathcal{F} \subset \mathcal{E}$ is a finite length coherent subsheaf with length λ, then $\phi(F^*\mathcal{F}) \subset \mathcal{E}$ is a finite length coherent subsheaf with length $p^d\lambda$. In particular, ϕ does not preserve $\mathcal{F}$ unless $d = 0$ or $\mathcal{F} = 0$.*

Proof. The case $d = 0$ is trivial, so we assume $d > 0$. As the claims are étale local on the base, we may assume that $X = \mathrm{Spec}(R)$ is local. In this case, the sheaf $\mathcal{F}$ admits a finite filtration whose graded pieces look like $R/\mathfrak{m}$ where $\mathfrak{m} \subset R$ is the maximal ideal. By flatness, we are reduced to verifying the claim for $\mathcal{F} \simeq R/\mathfrak{m}$. In this case, the sheaf $F^*(\mathcal{F})$ is simply $R/\mathfrak{m}^{[p]}$, where $\mathfrak{m}^{[p]}$ is the ideal generated by the p-th powers of elements of $\mathfrak{m}$. In particular, its length is clearly bigger than 1 as long as $d > 0$, proving the claim. ☐

Remark 3.2.3. The second proof of Proposition 3.2.1 given above uses the fact that R is Gorenstein, and that the top dimensional local cohomology group $H^d_{\mathfrak{m}}(R)$ does not contain any F-stable finite length submodules except 0. By [Smi97b, Theorem 2.6], it follows that proof goes through to show that any excellent local Gorenstein F-rational ring satisfies Condition 1.0.2. This proof will be re-explained in Chapter 7 with more context.

3.2.2 Some further affine examples

We now discuss a few further examples and one non-example of rings satisfying Condition 1.0.2. The key feature in the examples is that the corresponding characteristic 0 objects have rational singularities. The first example we look at is the cone on the standard quadric.

Example 3.2.4. We claim that $R = k[\![x_1, \dots, x_n]\!]/(\sum_i x_i^2)$ satisfies Condition 1.0.2 for $n \geq 3$ provided $\mathrm{char}(k) > 2$. By Remark 3.2.3, it suffices to show that R is F-rational. By [Hun96, Theorem 4.2], it suffices to show that $R/(x_n)$ is F-rational. Thus, we can set up an induction once we settle the $n = 3$ case. This case follows from [Hoc73, Example 3]. Alternately, in the $n = 3$ case, we may identify S with (completion at the origin of) the affine cone on a smooth conic in $\mathbf{P}^2$. Applying the techniques of Lemma 4.2.2 and Example 4.2.4, we compute that

$$H^2_{\mathfrak{m}}(R) \simeq \oplus_{n \in \mathbf{Z}} H^1(\mathbf{P}^1, \mathcal{O}(2n)).$$

The preceding presentation is Frobenius equivariant, where Frobenius acts on the grading on the right by multiplying the weights by p. By inspection, it easily follows then that $H^2_{\mathfrak{m}}(R)$ has no Frobenius-stable proper non-zero submodules. As R is Gorenstein we can follow the arguments given in the second proof of Theorem 3.2.1 to see that R satisfies Condition 1.0.2.

Next, we show that certain quotient singularities satisfy Condition 1.0.2.

Example 3.2.5. Let k be a field, and let R be a regular k-algebra. Let G be a linearly reductive group acting on R. Then $\mathrm{Spec}(R^G)$ satisfies Condition 1.0.2. Indeed, the inclusion $R^G \to R$ has an R^G-linear section by the Reynolds operator. Lemma 3.1.3 and Example 3.2.1 finish the proof. More generally, the same argument shows thay any subring A of a regular ring R that splits off as an A-linear summand satisfies Condition 1.0.2. In particular, if G is a reductive group over $\mathbf{C}$ acting on an affine algebraic $\mathbf{C}$-scheme $\mathrm{Spec}(R)$, then infinitely many positive characteristic reductions of $\mathrm{Spec}(R^G)$ satisfy Condition 1.0.2.

Lastly, we discuss a non-example due to Hochster: a hypersurface singularity of dimension 2 in characteristic 2 that violates Condition 1.0.2. Aside from its intrinsic interest, this example is meant to caution the reader against putting excessive faith in the belief that Condition 1.0.2 is equivalent to rationality: the standard lift of this hypersurface to characteristic 0 has rational singularities.

Example 3.2.6. Let k be a field of characteristic 2. Let $S = k[u, v]$ be a polynomial ring, and let $R = k[u^2, v^2, u^3 + v^3] \hookrightarrow S$. Since $\mathrm{char}(k) = 2$, R admits the presentation $R = k[x, y, z]/(x^3 + y^3 + z^2)$ where $x = u^2$, $y = v^2$, and $z = u^3 + v^3$. In particular, $\mathrm{Spec}(R)$ is a hypersurface singularity of dimension 2. Since the singularity is isolated, R is even normal. On the other hand, $\mathrm{Spec}(R)$ violates Condition 1.0.2 because the natural map $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a finite surjective map such that $\mathcal{O}_{\mathrm{Spec}(R)} \to f_*\mathcal{O}_{\mathrm{Spec}(S)}$ has no section: identifying sheaves with modules and applying such a section s to $u^3 + v^3 = u \cdot u^2 + v \cdot v^2$ would give us $u^3 + v^3 = s(u^3 + v^3) = s(u)u^2 + s(v)v^2 \in (u^2, v^2)R$ which is false. The same example can be adapted to arbitrary positive characteristic p by setting $R = k[u^p, v^p, u^a + v^a]$ for some $p < a < 2p$.

3.2.3 Some global examples

The examples discussed hitherto were all affine. Requiring Condition 1.0.2 on a projective variety X over a positive characteristic field k leads to questions of a very different flavour as the geometry of X is heavily constrained. For example, Theorem 5.0.1 from Chapter 5 shows that $H^i(X, \mathcal{O}) = 0$ for all $i > 0$. In fact, Theorem 5.0.1 implies that Condition 1.0.2 is equivalent to the much stronger sounding Condition 1.0.1. Thus, examples might be harder to find; nevertheless, they do exist. We will discuss such examples in §4.2.3 and §5.6. In the present section, we simply give an explicit example to show that not all smooth projective varieties satisfy Condition 1.0.2.

Example 3.2.7. Let E be an elliptic curve over an algebraically closed field k of positive characteristic. We will show that E does not satisfy Condition 1.0.2. In the case E is supersingular, we know that the Frobenius morphism $F : E \to E$ induces the 0 map $H^1(E, \mathcal{O}_E) \to H^1(E, \mathcal{O}_E)$. Identifying the target with the cohomology of the second term of the exact sequence

$$0 \to \mathcal{O}_E \to F_*\mathcal{O}_E \to \mathcal{Q} \to 0$$

shows that the map $\mathcal{O}_E \to F_*\mathcal{O}_E$ is not split. Thus, supersingular elliptic curves fail Condition 1.0.2. If E is ordinary, then $H^1(E, \mathbf{Z}/p) \neq 0$. The Artin-Schreier exact sequence of étale sheaves for E takes the form

$$0 \to \mathbf{Z}/p \to \mathcal{O}_E \stackrel{x^p - x}{\to} \mathcal{O}_E \to 0.$$

The map $x \mapsto x^p - x$ is clearly surjective on $k \simeq H^0(E, \mathcal{O}_E)$. It follows that the image of $H^1(E, \mathbf{Z}/p) \to H^1(E, \mathcal{O}_E)$ is non-zero. On the other hand, classes in $H^1(E, \mathbf{Z}/p)$ can be trivialised by finite covers. Indeed, a class is classified by a $\mathbf{Z}/p$-torsor $f : C \to E$, and passing to C kills the class. It follows then that the map $H^1(E, \mathcal{O}_E) \to H^1(C, \mathcal{O}_C)$ is 0. Therefore, the induced map $\mathcal{O}_E \to f_*\mathcal{O}_C$ is not split.

Chapter 4

The derived direct summand condition

In this chapter, we discuss the derived direct summand condition. Recall that this condition says:

Condition (1.0.1)**.** *Given a proper surjective morphism $f : X \to S$, the morphism $\mathcal{O}_S \to \mathrm{R}f_*\mathcal{O}_X$ admits a section in the derived category $\mathrm{D}(\mathrm{Coh}(S))$ coherent sheaves on S.*

The plan for this chapter is as follows: in §4.1 we discuss the equivalence of Condition 1.0.1 with that of rational singularities in characteristic 0 (due to Kovács [Kov00]), and in §4.2 we discuss some examples in charactersitic p which illustrate the parallels between Condition 1.0.1 and the notion of rational singularities.

4.1 The situation in characteristic 0

Condition 1.0.1 is closely related to the notion of rational singularities. Recall the following classical definition:

Definition 4.1.1. An noetherian integral scheme S has *rational singularities* if there exists a proper birational map $f : X \to S$ with X regular such that $\mathcal{O}_S \simeq \mathrm{R}f_*\mathcal{O}_X$.

Examples of rational singularities are cones on Fano hypersurfaces and quotient singularities. Kovács (see [Kov00]) shows that a complex variety S satisfies Condition 1.0.1 if and only if it has rational singularities. We review a proof below. Both directions are proven using Grothendieck duality and the Grauert-Riemenschneider vanishing theorems, following Kovács strategy for the forward direction. Kovács proves the reverse directon differently, using Kollár's vanishing theorems (see [Kol86a] and [Kol86b]). Consequently, he can prove more such as a higher dimensional version of Kempf's criterion for rational singularities. We eschew this perspective to keep our treatment low brow; the reader interested in the deeper proof is referred to [Kov00, Theorems 2 and 3].

Warning 4.1.2. Throughout this thesis, we often make statements to the effect that certain diagrams of complexes of exist and are commutative. In each such case, what is really meant is the corresponding statement in the derived category, i.e., that the relevant maps exist up to quasi-isomorphisms, and the diagrams commute up to homotopy. We have preferred to commit this abuse of language as we believe it improves readability without making any real sacrifices.

Theorem 4.1.3 (Kovács)**.** *Let S be a scheme of finite type over a field k of characteristic 0. Then S satisfies Condition 1.0.1 if and only if it has rational singularities.*

Proof. Let us prove first that if S satisfies Condition 1.0.1, then S has rational singularities. By Proposition 3.1.5, we may assume S is normal. By Hironaka's theorem (see [Hir64]) or even the weaker results in Abramovich-de Jong (see [AdJ97]), we may assume that there exists a proper birational map $f : X \to S$ with X smooth. The natural map $\mathcal{O}_S \to \mathrm{R}f_*\mathcal{O}_X$ has a section by assumption. Thus, we have a diagram

$$\mathcal{O}_S \to \mathrm{R}f_*\mathcal{O}_X \to \mathcal{O}_S$$

with the composite map the identity. Applying $\mathrm{R}\mathcal{H}\mathrm{om}(-, \omega_S^\bullet)$ with $\omega_S^\bullet$ the dualising complex on S (normalised so that the dualising sheaf sits in homological degree d), we obtain a diagram

$$\omega_S^\bullet \to \mathrm{R}\mathcal{H}\mathrm{om}(\mathrm{R}f_*\mathcal{O}_X, \omega_S^\bullet) \to \omega_S^\bullet$$

with the composite map the identity. By Grothendieck duality, the middle term is identified with $\mathrm{R}f_*\omega_X^\bullet$ where $\omega_X^\bullet$ is the dualising complex on X normalised as above. Thus, we obtain a diagram

$$\omega_S^\bullet \to \mathrm{R}f_*\omega_X^\bullet \to \omega_S^\bullet$$

with the composite map the identity. As X is smooth, $\omega_X^\bullet \simeq \omega_X[d]$ where $\omega_X = \det(\Omega_X^1)$ is the canonical bundle, and $d = \dim(X) = \dim(S)$. Grauert-Riemenschneider vanishing (see [Laz04a, Theorem 4.3.9]) tells us that $\mathrm{R}f_*\omega_X$ is concentrated in degree 0. Thus, the complex $\omega_S^\bullet$ is also concentrated in degree d. In particular, S is Cohen-Macaulay with dualising complex $\omega_S[d]$, where ω_S is the dualising sheaf. Moreover, the preceding diagram tells us that we have a diagram

$$\omega_S \to f_*\omega_X \to \omega_S$$

with the composite map the identity. As ω_X is a torsion free sheaf of generic rank 1, the same is true for $f_*\omega_X$. In particular, it admits no non-trivial direct summands for rank reasons. Hence, we have $\omega_S \simeq f_*\omega_X$ and, therefore, $\omega_S^\bullet \simeq \mathrm{R}f_*\omega_X^\bullet$. Now we have the following sequence of canonical isomorphisms:

$$\mathcal{O}_S \simeq \mathrm{R}\mathcal{H}\mathrm{om}(\omega_S^\bullet, \omega_S^\bullet) \simeq \mathrm{R}\mathcal{H}\mathrm{om}(\mathrm{R}f_*\omega_X^\bullet, \omega_S^\bullet) \simeq \mathrm{R}f_*\mathrm{R}\mathcal{H}\mathrm{om}(\omega_X^\bullet, \omega_X^\bullet) \simeq \mathrm{R}f_*\mathcal{O}_X$$

which implies that S has rational singularities.

For the reverse implication, let S be a complex variety with rational singularities, i.e., there exists a resolution $f : X \to S$ such that $\mathcal{O}_S \simeq \mathrm{R}f_*\mathcal{O}_X$. Let $g : Y \to S$ be a proper surjective morphism. We need to show that $\mathcal{O}_S \to \mathrm{R}g_*\mathcal{O}_Y$ has a section. By repeatedly cutting Y by hyperplane sections, we may assume that g is generically finite. By the Raynaud-Gruson flattening theorem (see [RG71, Théorème 5.2.2]) or a simple Hilbert scheme argument, we can find a diagram as follows:

$$\begin{array}{ccc} Y'' & \longrightarrow & Y \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow{b} & S. \end{array}$$

Here b is a normalised blowup of S, h is the strict transform of g along b, and h is finite flat. As we are in characteristic 0 and Y' is normal, $\mathcal{O}_{Y'} \to h_*\mathcal{O}_{Y''} \simeq \mathrm{R}h_*\mathcal{O}_{Y''}$ has a section coming from the trace map. Thus, to show that $\mathcal{O}_S \to \mathrm{R}g_*\mathcal{O}_Y$ has a section, it suffices to show that $\mathcal{O}_S \to \mathrm{R}b_*\mathcal{O}_{Y'}$ has a section. In other words, we may assume that g is a modification. Moreoever, by Hironaka or Abramovich-de Jong as above, we may even assume that Y is smooth. The situation now can be summarised in the following diagram:

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & S. \end{array}$$

Here f is the original resolution witnessing S having rational singularities, while g is another resolution. We need to show that $\mathcal{O}_S \to \mathrm{R}g_*\mathcal{O}_Y$ has a section. Using resolutions, we may fill up the above diagram to get a diagram

$$\begin{array}{ccc} Z & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & S \end{array}$$

where all maps are proper birational maps, and Z is smooth. We will first show that the map $\mathcal{O}_X \to \mathrm{R}g'_*\mathcal{O}_Z$ has a section, and then use this section to get the desired result.

The pushforwards $\mathrm{R}^i g'_*\omega_Z$ vanish for $i > 0$ by Grauert-Riemenschneider vanishing (see [Laz04a, Theorem 4.3.9]). By birational invariance of plurigenera, we also know that the trace map $H^0(Z, \omega_X) \simeq H^0(X, g'_*\omega_Z) \to H^0(X, \omega_X)$ induces an isomorphism. Hence, $g'_*\omega_Z \simeq \omega_X$, and consequently, $\mathrm{R}g'_*\omega_Z^\bullet \simeq \omega_X^\bullet$. Applying $\mathrm{R}\mathcal{H}\mathrm{om}(-, \omega_S^\bullet)$, we find that $\mathcal{O}_X \simeq \mathrm{R}g'_*\mathcal{O}_Z$ via the natural map. By the commutativity of the square

$$\begin{array}{ccc} \mathrm{R}(g_* \circ f'_*)\mathcal{O}_Z \simeq \mathrm{R}(f_* \circ g'_*)\mathcal{O}_X & \longleftarrow & \mathrm{R}f_*\mathcal{O}_X \\ \uparrow & & \uparrow \\ \mathrm{R}g_*\mathcal{O}_Y & \longleftarrow & \mathcal{O}_S \end{array}$$

and the fact the top and right arrows have sections, it follows that $\mathcal{O}_S \to \mathrm{R}g_*\mathcal{O}_Y$ has a section. ☐

Remark 4.1.4. Theorem 4.1.3 uses the characteristic 0 assumption in each direction of the implication. Namely, both the Grauert-Riemenschneider vanishing theorem and resolution of singularities are used in each direction. We do not know which direction, if any, survives in general.

4.2 The situation in characteristic p

As indicated in Remark 4.1.4, we do not know the exact relation between Condition 1.0.1 and the theory of rational singularities. Nevertheless, we believe that the two are very closely related. In this section, we will provide some examples and non-examples to support this intuition.

4.2.1 Regular schemes

We first discuss the example of regular schemes. This can be viewed as the positive characteristic analogue of the fact that smooth varieties have rational singularities. The bulk of that proof will be explained in Chapter 5.

Theorem 4.2.1. *Let $S = \mathrm{Spec}(R)$ with R be a regular $\mathbf{F}_p$-algebra. Then S satisfies Condition 1.0.1.*

Proof. Let $f : X \to S$ be a proper surjective morphism. By Theorem 5.0.1, there exists a finite surjective morphism $\pi : Y \to X$ such that, with $g = f \circ \pi$, the pullback map $\tau_{\geq 1}\mathrm{R}f_*\mathcal{O}_X \to \tau_{\geq 1}\mathrm{R}g_*\mathcal{O}_Y$ is 0. By applying $\mathrm{Hom}(\mathrm{R}f_*\mathcal{O}_X, -)$ to the exact triangle

$$g_*\mathcal{O}_Y \to \mathrm{R}g_*\mathcal{O}_Y \to \tau_{\geq 1}\mathrm{R}g_*\mathcal{O}_Y \to g_*\mathcal{O}_Y[1]$$

we see that the natural pullback map $\mathrm{R}f_*\mathcal{O}_X \to \mathrm{R}g_*\mathcal{O}_Y$ factors through $g_*\mathcal{O}_Y \to \mathrm{R}g_*\mathcal{O}_Y$. As $g : Y \to S$ is a proper surjective morphism, the algebra $g_*\mathcal{O}_Y$ is a coherent sheaf of algebras corresponding to the structure sheaf of a finite surjective morphism. By Theorem 3.2.1, the natural map $\mathcal{O}_S \to g_*\mathcal{O}_Y$ has a splitting, and thus the same is true for $\mathcal{O}_S \to \mathrm{R}f_*\mathcal{O}_X$. □

4.2.2 Some cones

We now discuss a non-example: a cone on an elliptic curve. Such singularities are not rational in characteristic 0 (this fact is well-known, follows from Theorem 4.1.3, and is explained below). In order to prove that this singularity violates Condition 1.0.1 in general, we need a result in algebra.

Lemma 4.2.2. *Let $X = \mathrm{Spec}(R)$ with $(R, \mathfrak{m})$ a Cohen-Macaulay local ring of dimension $d > 1$ essentially of finite type over a field k, and let $j : V \hookrightarrow X$ be complement of the closed point. Then we have:*

1. $\mathrm{R}^0 j_*\mathcal{O}_V \simeq R$.
2. $\mathrm{R}^i j_*\mathcal{O}_V = 0$ *for* $0 < i < d - 1$.
3. $\mathrm{R}^{d-1} j_*\mathcal{O}_V \simeq H^d_{\mathfrak{m}}(R)$.
4. *The R-module $\mathrm{Ext}^d_R(H^d_{\mathfrak{m}}(R), R)$ is free of rank 1. The complex $\mathrm{R}j_*\mathcal{O}_V$ determines a canonical generator of this module.*

Proof. For any local scheme $(X, 0) = (\mathrm{Spec}(R), \mathfrak{m})$, with $U = X - \{x\}$, we have an exact triangle

$$\mathrm{R}\Gamma_{\mathfrak{m}}(R) \to \mathrm{R}\Gamma(X, \mathcal{O}_X) \to \mathrm{R}\Gamma(U, \mathcal{O}_U) \to \mathrm{R}\Gamma_{\mathfrak{m}}(R)[1].$$

Since X is affine, the middle term vanishes in positive degrees. The first three claims now follow as the Cohen-Macaulay condition is equivalent to the vanishing of $H^i_{\mathfrak{m}}(R)$ for $i < d$. For the last claim, note that everything can be checked after passage to the algebraic closure. Thus, we assume that k is algebraically closed. We will first verify the claim for R regular, and then deduce the one for the Cohen-Macaulay case by flat base change.

Assume that R is regular. As all the claims being made are local on $\mathrm{Spec}(R)$ and detectable after completion, we reduce to the case that R is complete. Since k is algebraically closed, there exists an isomorphism $k[\![x_1, \cdots, x_d]\!] \simeq R$. Since completion commutes with taking local cohomology, we reduce our calculations to understanding the local cohomology of $\mathbf{A}^d$ at the origin. Using the $\mathbf{G}_m$-action, we find an $H^0(\mathbf{A}^d - \{0\}, \mathcal{O})$-equivariant identification

$$\mathrm{R}^i j_*\mathcal{O}_V \simeq \oplus_{k\in\mathbf{Z}} H^i(\mathbf{P}^{d-1}, \mathcal{O}(k)).$$

Now the claim follows from standard calculations of the cohomology of projective space.

In the general Cohen-Macaulay case, we localise as above and assume that there exists a finite morphism $f : A \to R$ with $(A, \mathfrak{m}_A)$ a regular local ring of dimension d. Since R is Cohen-Macaulay, the map f is finite flat thanks to Auslander-Buschbaum. If we let $U = \mathrm{Spec}(A) - \{\mathfrak{m}_A\}$, then flat base change (see [Gro61, Proposition 1.4.15]) gives us an isomorphism

$$a : \mathrm{R}\Gamma(V, \mathcal{O}_V) \simeq \mathrm{R}\Gamma(U, \mathcal{O}_U) \otimes_A R.$$

Given the claims for A, it now suffices to verify that $\mathrm{Ext}^d_R(H^d_{\mathfrak{m}}(R), R)$ is free of rank 1. By the flatness of f, for any A-module M, we have a change of rings isomorphism

$$\mathrm{Ext}^p_R(M \otimes_A R, R) \simeq \mathrm{Ext}^p_A(M, R).$$

In the case that $M = H^d_{\mathfrak{m}_A}(A)$, we know that $M \otimes_A R \simeq H^d_{\mathfrak{m}}(R)$ by the flat base change isomorphism a. Thus, the above isomorphism gives

$$\mathrm{Ext}^d_R(H^d_{\mathfrak{m}}(R), R) \simeq \mathrm{Ext}^d_A(H^d_{\mathfrak{m}}(A), R) \simeq \mathrm{Ext}^d_A(H^d_{\mathfrak{m}}(A), A) \otimes R \simeq A \otimes R \simeq R.$$

Hence, the Ext-group has the correct form, as desired. □

Remark 4.2.3. It is probably unnecessary to assume that we are working over a field in Lemma 4.2.2. The natural approach to proving this would be to use the Cohen presentation theorems and proceed as above. We do not pursue this extra generality here as it is not needed for Example 4.2.4.

Here is the promised example.

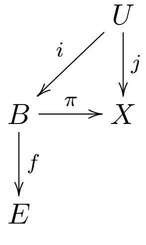
Example 4.2.4. Let k be a field, and let $f(x, y, z)$ be a homogeneous cubic defining a non-singular elliptic curve $E \subset \mathbf{P}^2$. Let $X = \mathrm{Spec}(k[x, y, z]/(f))$ be the affine cone on E. We will show that X does not satisfy Condition 1.0.1. To show this, it suffices to exhibit a proper surjective morphism $\pi : B \to X$ such that $\mathcal{O}_X \to \mathrm{R}\pi_*\mathcal{O}_B$ is not split. Our choice of B will be the blowup of X at the origin. The normality of X and the fact that the blowup map $\pi : B \to X$ is an isomorphism away from the origin imply that $\mathcal{O}_X \simeq \mathrm{R}^0\pi_*\mathcal{O}_B$. As the fibres of π have dimension at most 1, it follows that $\mathrm{R}^i\pi_*\mathcal{O}_B = 0$ for $i > 1$. We will show that $\mathrm{R}^1\pi_*\mathcal{O}_X \simeq \kappa(0)$ where $\kappa(0)$ is the structure sheaf of the origin, and that the extension class in $\mathrm{Ext}^2_X(\kappa(0), \mathcal{O}_X)$ determined by the complex $\mathrm{R}\pi_*\mathcal{O}_B$ is non-zero. This will suffice to prove the claim, as the existence of a section of $\mathcal{O}_X \to \mathrm{R}\pi_*\mathcal{O}_B$ is tantamount to splitting the complex $\mathrm{R}\pi_*\mathcal{O}_B$.

The space B can be identified with the total space of the line bundle $\mathcal{O}(-1)$ on E with the zero section coinciding with the exceptional divisor. If $f : B \to E$ denotes the projection map, then the affineness of f and the construction of B imply that

$$\mathrm{R}f_*\mathcal{O}_B = f_*\mathcal{O}_B = \oplus_{i \geq 0}\mathcal{O}(i).$$

Riemann-Roch then shows that $H^1(E, \mathcal{O}) \simeq H^1(E, \mathrm{R}f_*\mathcal{O}_B) \simeq H^1(B, \mathcal{O})$. As X is affine, it follows that $H^0(\mathrm{R}^1\pi_*\mathcal{O}_B)$ is 1-dimensional. The sheaf $\mathrm{R}^1\pi_*\mathcal{O}_B$ is only supported at the origin, so $\mathrm{R}^1\pi_*\mathcal{O}_B \cong \kappa(0)$.

Let $U \subset X$ be the complement of the origin in X. As $\pi : B \to X$ is an isomorphism over U, we may identify U with an open subset of B. We may summarise the picture obtained thus far in the following diagram:



Under the identification of B with the total space of the line bundle $\mathcal{O}(-1)|_E$, the open subset U is identified with the complement of the zero section, i.e., as the total space of the $\mathbf{G}_m$-torsor $\mathcal{O}(-1)|_E - 0(E)$ over E. This tells us that

$$\mathrm{R}(\pi \circ i)_* \mathcal{O}_U = \pi_* i_* \mathcal{O}_U = \oplus_{i \in \mathbf{Z}} \mathcal{O}(i).$$

Hence, $H^1(U, \mathcal{O}) = \oplus_{i \in \mathbf{Z}} H^1(E, \mathcal{O}(i))$.

The factorisation $j = \pi \circ i$ gives us a morphism $i^* : \mathrm{R}\pi_* \mathcal{O}_B \to \mathrm{R}j_* \mathcal{O}_U$. Identifying sheaves with their spaces of sections and using the previous calculations, we see that i^* is an isomorphism on R^0, and induces the inclusion of a 1-dimensional vector space $H^1(B, \mathcal{O})$, viewed as a sheaf on X supported at the origin, into the R-module $H^1(U, \mathcal{O})$. By Lemma 4.2.2, we may identify $H^1(U, \mathcal{O}) = H^0(\mathrm{R}^1 j_* \mathcal{O}_U)$ with the local cohomology group $H^2_{\mathfrak{m}}(R)$ which, by the Gorenstein property, is identified with an injective hull of the residue field k. Thus, the map on R^1 induced by i^* can be identified with an injective map $a : k \subset H^2_{\mathfrak{m}}(R)$, which is unique up to scaling because $H^2_{\mathfrak{m}}(R)$ is an injective hull of k. We may summarise this information in the following morphism of exact triangles

$$\begin{array}{ccccccc}
\mathcal{O}_X & \longrightarrow & \mathrm{R}\pi_* \mathcal{O}_B & \longrightarrow & H^1(B, \mathcal{O})[-1] \simeq k[-1] & \xrightarrow{\mathrm{ob}(B)} & \mathcal{O}_X[1] \\
\| & & \downarrow & & \downarrow{\scriptstyle a[-1]} & & \| \\
\mathcal{O}_X & \longrightarrow & \mathrm{R}j_* \mathcal{O}_U & \longrightarrow & H^1(U, \mathcal{O})[-1] \simeq H^2_{\mathfrak{m}}(R)[-1] & \xrightarrow{\mathrm{ob}(U)} & \mathcal{O}_X[1].
\end{array}$$

By Lemma 4.2.2, the class $\mathrm{ob}(U)$ determines a canonical generator of the group $\mathrm{Ext}^2_R(H^2_{\mathfrak{m}}(R), R) \simeq R$. Since R is Gorenstein, the functor $\mathrm{Ext}^2_R(-, R)$ is simply the local duality functor. By the discussion above, the map a was injective. Thus, by duality, the map $\mathrm{Ext}^2_R(H^2_{\mathfrak{m}}(R), R) \to \mathrm{Ext}^2_R(k, R)$ determined by a is simply the natural projection map $R \to R/\mathfrak{m}$ (up to a unit). In particular, the image $\mathrm{ob}(B)$ of the class $\mathrm{ob}(U)$ is non-zero, as desired.

Remark 4.2.5. The methods of Example 4.2.4 can be adapted to show the following more general statement: if $(S, 0) = (\mathrm{Spec}(R), \mathfrak{m})$ is a Gorenstein normal surface singularity, and $f : X \to S$ is a resolution with $\mathrm{R}^1 f_* \mathcal{O}_X \neq 0$, then $\mathcal{O}_S \to \mathrm{R}f_* \mathcal{O}_X$ does not admit a section. In order to apply the arguments of Example 4.2.4, one needs to know that the map $i^* : \mathrm{R}^1 f_* \mathcal{O}_X \to \mathrm{R}^1 j_* \mathcal{O}_U$ is non-zero, where $j : U \hookrightarrow S$ is the punctured spectrum, and $i : U \to X$ is a section of f over U. To see this, let E denote the exceptional fibre $f^{-1}(0)$. Note that $E = X - i(U)$. Since E is a Cartier divisor, the functor i_* is exact. Now consider the short exact sequence

$$0 \to \mathcal{O}_X \to i_* \mathcal{O}_U \to \mathcal{Q} \to 0$$

Since i_* is exact, the middle term computes the cohomology of U. Hence, our claim will follow if we can show that $H^0(\mathcal{Q}) = 0$. The sheaf $\mathcal{Q}$ has a natural presentation

$$\mathcal{Q} = \mathrm{colim}_n \, \mathcal{O}_X(nE)/\mathcal{O}_X$$

which defines a filtration whose associated graded pieces look like $\mathcal{O}_X(nE) \otimes \mathcal{O}_E$ with $n > 0$. Since E is exceptional, we know that $\mathcal{O}_X(nE) \otimes \mathcal{O}_E$ is an antiample line bundle on E when $n > 0$. It follows that $H^0(\mathcal{O}_X(nE) \otimes \mathcal{O}_E) = 0$ when $n > 0$. By devissage and the commutation of cohomology with inductive limits, it follows that $H^0(\mathcal{Q}) = 0$, as desired.

4.2.3 Some global examples

Our goal in this section is to discuss some *global* or *projective* examples of varieties satisfying Condition 1.0.1, fulfilling a promise made in §3.2.3. The techniques used in Theorem 4.1.3 show that Condition 1.0.1 is a *local* condition in characteristic 0. In particular, all smooth projective varieties over $\mathbf{C}$ satisfy it. On the other hand, as explained in Example 3.2.7, even Condition 1.0.2 is not a local condition in positive characteristic. In fact, thanks to Theorems 5.0.1 and 5.0.2, we know that projective varieties S satisfying Condition 1.0.1 in positive characteristic are strongly constrained: the groups $H^i(S, \mathcal{O})$ vanish for $i > 0$, for example. Nevertheless, we will show below that there are large classes of varieties satisfying Condition 1.0.1, namely, the toric ones. In §5.6, we will give a non-toric exampple.

We begin with some preliminary lemmas. The first one is an analogue of Lemma 3.1.3 with an identical proof.

Lemma 4.2.6. *Let $\pi : U \to X$ be a morphism such that $\mathcal{O}_X \to \mathrm{R}\pi_*\mathcal{O}_U$ has a section. Then X satisfies Condition 1.0.1 if U does so.*

Proof. Let $g : Z \to X$ be a proper surjective morphism. Consider the diagram

$$\begin{array}{ccc} Z_U = Z \times_X U & \xrightarrow{a} & U \\ \downarrow b & & \downarrow \pi \\ Z & \xrightarrow{g} & X \end{array}$$

The maps a and b are defined by the diagram. This square gives rise to the following commutative square in $\mathrm{D}(\mathrm{Coh}(X))$:

$$\begin{array}{ccc} \mathrm{R}(g \circ b)_*\mathcal{O}_{Z_U} \simeq \mathrm{R}(\pi \circ a)_*\mathcal{O}_{Z_U} & \longleftarrow & \mathrm{R}\pi_*\mathcal{O}_U \\ \uparrow & & \uparrow \\ \mathrm{R}g_*\mathcal{O}_Z & \longleftarrow & \mathcal{O}_X \end{array}$$

The arrows on the right and the top are split by assumption. It follows formally that the same is true for the bottom arrow, as desired. ☐

Next, we show that Condition 1.0.1 restricts well to open subschemes.

Lemma 4.2.7. *Let $U \hookrightarrow S$ be an open immersion of noetherian schemes. Then U satisfies Condition 1.0.1 if S does so.*

Proof. Let $f : Y \to U$ be a proper surjective morphism. By Nagata compactification, we can find an extension $\overline{f} : \overline{Y} \to S$ of f to S. By assumption, we know that $\mathcal{O}_S \to \mathrm{R}\overline{f}_*\mathcal{O}_{\overline{Y}}$ is split. Restricting such a section to U and using the commutation of cohomology with flat base change gives the desired result for U. ☐

As a corollary, we arrive at the desired examples.

Corollary 4.2.8. *Toric varieties that are projective over an affine satisfy Condition 1.0.1. In particular, projective spaces and their products satisfy Condition 1.0.1.*

Proof. A toric variety X that is projective over an affine can be obtained as a quotient U/G (see [MS05, Theorem 10.27]), where $U \subset \mathbf{A}^n$ is an open subscheme, and $G \subset \mathbf{G}_m^n$ is an algebraic subgroup preserving U. As $\mathbf{G}_m^n$ is linearly reductive, so is G (see [AOV08, Proposition 2.5]). In particular, we see that $\mathcal{O}_X \to \pi_*\mathcal{O}_U \simeq \mathrm{R}\pi_*\mathcal{O}_U$ is a direct summand. The result now follows by combining Lemmas 4.2.6 and 4.2.7 and the fact that $\mathbf{A}^n$ satisfies Condition 1.0.2 thanks to Hochster's theorem (see Proposition 3.2.1). □

Chapter 5

Positive characteristic

This chapter forms the core of our investigations into the nature of Conditions 1.0.2 and 1.0.1 in positive characteristic p. The main goal of this chapter is to prove the following two theorems:

Theorem 5.0.1. *Let $f : X \to S$ be a proper morphism of noetherian $\mathbf{F}_p$-schemes of finite Krull dimension. Then there exists a finite surjective morphism $\pi : Y \to X$ such that, with $g = f \circ \pi$, the pullback map $\pi^* : \tau_{\geq 1}\mathrm{R}f_*\mathcal{O}_X \to \tau_{\geq 1}\mathrm{R}g_*\mathcal{O}_Y$ is 0.*

Theorem 5.0.2. *Given a noetherian $\mathbf{F}_p$-scheme S, Condition 1.0.2 is satisfied if and only if Condition 1.0.1 is satisfied.*

This chapter is organised as follows: in §5.1 we review some general results about derived categories used in our proof; in §5.2 we prove Theorems 5.0.1 and 5.0.2; in §5.3 we make some comments related to our proof of Theorems 5.0.1 and 5.0.2; in §5.4, we formulate and prove some (essentially already known) results in commutative algebra using the above theorems; in §5.5 we use our results to answer some questions raised by Karen Smith and also provide some counterexamples that seem to be missed in the literature; in §5.6 we use our results to provide some examples of non-toric smooth projective varieties satisfying Condition 1.0.2.

5.1 Some facts about derived categories

The purpose of this section is to collect a couple of well-known facts about triangulated categories for later use. As a general reference for triangulated categories and t-structures, we suggest [BBD82]. For the convenience of the reader, we recall some notation regarding truncations

Notation 5.1.1. Let $\mathcal{D}$ be a triangulated category with a t-structure given by a pair $(\mathcal{D}^{\geq 0}, \mathcal{D}^{\leq 0})$ of full subcategories satisfying the usual axioms. For each integer n, we let $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$ (respectively, $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$); this can be thought of as the fullsubcategory spanned by objects with cohomology only in degree at least (respectively, at most) n. Moreover, there exist truncation functors: for each integer n, there exist endofunctors $\tau_{\leq n}$ and $\tau_{\geq n}$ of $\mathcal{D}$ which are retractions of $\mathcal{D}$ onto the fullsubcategories $\mathcal{D}^{\leq n}$ and $\mathcal{D}^{\geq n}$. We let $\tau_{>n} = \tau_{\geq n+1}$, and $\tau_{<n} = \tau_{\leq n-1}$. These truncation functors are not exact, but they sit in an exact triangle

$$\tau_{\leq n} \to \mathrm{id} \to \tau_{>n} \to \tau_{\leq n}[1]$$

Lastly, we point out that the preceding notation clashes with the standard notation from topology concerning Postnikov truncation functors: if A is a complex of abelian groups bounded below 0, and K denotes the Dold-Kan functor which takes such a complex to its associated simplicial set, then $K(\tau_{\geq -n}(A)) \simeq \tau_{\leq n}K(A)$, where the latter is the Postnikov truncation of $K(A)$ in homotopy $\leq n$. This will not be an issue for us as we will never work with triangulated categories not coming from an abelian category.

5.1.1 Cohomologically trivial maps are nilpotent

Fix a triangulated category $\mathcal{D}$, with a t-structure $(\mathcal{D}^{\geq 0}, \mathcal{D}^{\leq 0})$. The first result we need arises from the following question: given a morphism $f : K \to L$ in $\mathcal{D}$ such that $H^*(f) = 0$, when can we conclude that $f = 0$?

As the non-trivial extension $\mathbf{Z}/2 \to \mathbf{Z}/2[1]$ in the derived category $\mathrm{D}(\mathrm{Ab})$ of abelian groups shows, the short answer is "not always". To understand this phenomenon better, fix a test object $M \in \mathcal{D}$, and consider the associated map of abelian groups

$$\mathrm{Hom}(M, f) : \mathrm{Hom}(M, K) \to \mathrm{Hom}(M, L)$$

The chosen t-structure gives rise to a functorial filtration on the morphism spaces of $\mathcal{D}$. Thus, the preceding map is a filtered map of filtered abelian groups. The assumption that $H^*(f) = 0$ implies that this filtered map induces the 0 map on the associated graded pieces. In other words, f moves the filtration one level down. This simple analysis suggests that under certain boundedness hypotheses, we may be able to salvage an implication of the form "$H^*(f) = 0 \Rightarrow f = 0$" at the expense of iterating a map like f a few times. This idea informs the title of this section, and is formalised in the following lemma:

Lemma 5.1.2. *Let $\mathcal{D}$ be a triangulated category with t-structure $(\mathcal{D}^{\geq 0}, \mathcal{D}^{\leq 0})$ whose heart is $\mathcal{A}$. Assume that for a fixed integer $d > 0$, we are given objects $K_1, \dots, K_{d+1} \in \mathcal{D}^{[1,d]}$ and maps $f_i : K_i \to K_{i+1}$ such that $H^{d+1-i}(f_i) = 0$ for all i. Then the composite map $f_d \circ \cdots \circ f_2 \circ f_1 : K_1 \to K_d$ is the 0 map.*

Proof. In $\mathcal{D}$, we have the exact triangle

$$\tau_{\leq d-1}K_2 \to K_2 \to H^d(K_2)[-d] \to \tau_{\leq d-1}K_2[1]$$

where $\tau_{\leq j} : \mathcal{D} \to \mathcal{D}^{\leq j}$ are the truncation functors associated to the given t-structure. Applying $\mathrm{Hom}_{\mathcal{D}}(K_1, -)$, using that it's a triangulated functor, and using the formula

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}(K_1, H^d(K_2)[-d]) &= \mathrm{Hom}_{\mathcal{D}}(K_1[d], H^d(K_2)[0]) \\ &= \mathrm{Hom}_{\mathcal{D}^{\leq 0}}(K_1[d], H^d(K_2)[0]) \\ &= \mathrm{Hom}_{\mathcal{D}^{\leq 0}}(H^0(K_1[d]), H^d(K_2)[0]) \\ &= \mathrm{Hom}_{\mathcal{D}^{\leq 0}}(H^d(K_1)[0], H^d(K_2)[0]) \\ &= \mathrm{Hom}_{\mathcal{A}}(H^d(K_1), H^d(K_2)) \end{aligned}$$

we see that the map $K_1 \to H^d(K_2)[-d]$ factors through $H^d(f_1)$ and, consequently by hypothesis, is 0. Thus, we obtain a (non-unique) factorisation of f_1 of the form $K_1 \to \tau_{\leq d-1}K_2 \to K_2$. The same method shows that the morphism $\tau_{\leq d-i}K_{i+1} \to \tau_{\leq d-i}K_{i+2}$ factors through $\tau_{\leq d-(i+1)}K_{i+2}$.

Thus, we obtain a diagram of morphisms:

$$\begin{array}{ccc}
K_1 & = & K_1 \\
\downarrow & & \downarrow {\scriptstyle f_1} \\
\tau_{\leq d-1} K_2 & \longrightarrow & K_2 \\
\downarrow & & \downarrow {\scriptstyle f_2} \\
\cdots & \longrightarrow & \cdots \\
\downarrow & & \downarrow {\scriptstyle f_d} \\
\tau_{\leq 0} K_{d+1} & \longrightarrow & K_{d+1}
\end{array}$$

As $K_{d+1} \in \mathrm{D}^{\geq 1}(\mathcal{A})$, we see that $\tau_{\leq 0} K_{d+1} = 0$. Thus, the composite vertical morphism on the left is zero, which implies that the one on the right is 0 as well. $\square$

Remark 5.1.3. It seems worthwhile to point out that Lemma 5.1.2 is proven in the abstract setting of triangulated categories with t-structures rather than the concrete setting of derived categories. In particular, it applies to triangulated categories of non-algebraic origin, such as the stable homotopy category of spaces.

5.1.2 Behaviour of truncations

We need a fact concerning a universal property of the truncation functors associated to a t-structure. The proof is completely trivial given the definition of a t-structure; the only reason we state it here is that we use it several times.

Lemma 5.1.4. *Let $\mathcal{D}$ be a triangulated category with t-structure $(\mathcal{D}^{\geq 0}, \mathcal{D}^{\leq 0})$ whose heart is $\mathcal{A}$. Given an object $K \in \mathcal{D}^{\geq 0}$, the natural transformation of functors* $\mathrm{Hom}(-, \tau_{\leq 0} K) \to \mathrm{Hom}(-, K)$ *is an isomorphism when restricted to $\mathcal{D}^{\leq 0}$. In particular, for any object $A \in \mathcal{A}$, we have an identification* $\mathrm{Hom}(A, \tau_{\leq 0} K) \simeq \mathrm{Hom}(A, K)$

Proof. As $K \in \mathcal{D}^{\geq 0}$, we have an exact triangle

$$\tau_{\geq 1} K[-1] \to H^0(K) \to K \to \tau_{\geq 1} K$$

with $\tau_{\geq 1} K \in \mathcal{D}^{>0}$ and, therefore, $\tau_{\geq 1} K[-1] \in \mathcal{D}^{>1} \subset \mathcal{D}^{>0}$. On other hand, by one of the axioms for a t-structure, the group $\mathrm{Hom}(\mathcal{D}^{\leq 0}, \mathcal{D}^{>0})$ vanishes. Fixing an object $L \in \mathcal{D}^{\leq 0}$ and applying the triangulated functor $\mathrm{Hom}(L, -)$ to the preceding triangle now finishes the proof. $\square$

Remark 5.1.5. Lemma 5.1.4 admits the following generalisation to homotopy theory: if X is an n-connected CW complex (i.e., $\pi_i(X) = 0$ for $i \leq n$) and Y is an $(n+1)$-truncated connected CW complex (i.e., $\pi_i(Y) = 0$ for $i > n+1$), then any map $f : X \to Y$ is uniquely determined (up to homotopy) by the induced map $\pi_{n+1}(X) \to \pi_{n+1}(Y)$. As the homotopy category of spaces in not a triangulated category, we cannot directly apply Lemma 5.1.4. Instead, one can argue as follows: as Y is $(n+1)$-truncated, the map $f : X \to Y$ uniquely factors as $X \to \tau_{\leq n+1}(X) \simeq K(\pi_{n+1}(X), n+1) \to Y$. The claim now boils down to verifying that maps $K(\pi, n+1) \to Y$ are uniquely determined by the induced map $K(\pi, n+1) \to K(\pi_{n+1}(Y), n+1)$ for any group π (abelian if $n+1 \geq 2$). This can be shown by climbing up the Postnikov tower of Y and observing that, by Hurewicz, classes in $H^i(K(\pi, n+1), \Lambda)$ vanish for $i \leq n$ and any coefficient system Λ.

5.2 The main theorem

This section is dedicated to the proof of Theorems 5.0.1 and 5.0.2. In fact, the bulk of the work involves proving Theorem 5.0.1 as Theorem 5.0.2 then follows by a fairly formal argument. The proof we give here draws on ideas whose origin can be traced back to Hochster and Huneke's work [HH92] on big Cohen-Macaulay algebras in positive characteristic. Later in this thesis, we provide two more proofs of this result: in §6.4, we reprove Theorem 5.0.1 using some general results about finite flat group schemes, while in §8.2, we prove a mixed characteristic enhancement of Theorem 5.0.1 by completely different geometric methods.

We begin with a rather elementary result on extending covers of schemes.

Proposition 5.2.1. *Fix a noetherian scheme X. Given an open dense subscheme $U \to X$ and a finite (surjective) morphism $f : V \to U$, there exists a finite (surjective) morphism $\overline{f} : \overline{V} \to X$ such that $\overline{f}_U$ is isomorphic to f. Given a Zariski open cover $\mathcal{U} = \{j_i : U_i \to X\}$ with a finite index set, and finite (surjective) morphisms $f_i : V_i \to U_i$, there exists a finite (surjective) morphism $f : Z \to X$ such that f_{U_i} factors through f_i. The same claims hold if "finite (surjective)" is replaced by "proper (surjective)" everywhere.*

Proof. We first explain how to deal with the claims for finite morphisms. For the first part, Zariski's main theorem (Théorème 8.12.6 of [Gro66]) applied to the morphism $V \to X$ gives a factorisation $V \hookrightarrow W \to X$ where $V \hookrightarrow W$ is an open immersion, and $W \to X$ is a finite morphism. The scheme-theoretic closure $\overline{V}$ of V in W provides the required compactification in view of the fact that finite morphisms are closed.

For the second part, by the first part, we may extend each $j_i \circ f_i : V_i \to X$ to a finite surjective morphism $\overline{f_i} : \overline{V_i} \to X$ such that $\overline{f_i}$ restricts to f_i over $U_i \hookrightarrow X$. Setting W to be the fibre product over X of all the $\overline{V_i}$ is then seen to solve the problem.

To deal with the case of proper (surjective) morphisms instead of finite (surjective) , we repeat the same argument as above replacing the reference to Zariski's main theorem by one to Nagata's compactification theorem (see [Con07, Theorem 4.1]). $\square$

Next, we present the primary ingredient in the present proof of Theorem 5.0.1: a general technique for constructing covers to annihilate coherent cohomology of $\mathbf{F}_p$-schemes under suitable finiteness assumptions. The method of construction is essentially borrowed from [HL07] where it is used to reinterpret and simplify the "Equational Lemma," one of the main ingrendients in the proof of the existence of big Cohen-Macaulay algebras in positive characteristic (see [HH92]).

Proposition 5.2.2. *Let X be a noetherian $\mathbf{F}_p$-scheme with $H^0(X, \mathcal{O}_X)$ finite over a ring A. Given an A-finite Frobenius-stable submodule $M \subset H^i(X, \mathcal{O}_X)$ for $i > 0$, there exists a finite surjective morphism $\pi : Y \to X$ such that $\pi^*(M) = 0$*

Proof. We first explain the idea informally. As M is A-finite, it suffices to work one cohomology class at a time. If $m \in M$, then the Frobenius-stability of M gives us a monic additive polynomial $g(X^p)$ such that $g(m) = 0$ where X^p acts by Frobenius. After adjoining g-th roots of certain local functions representing a coboundary, we can promote the preceding equation in cohomology to an equation of cocycles, i.e, we find $g(\overline{m}) = 0$ where $\overline{m}$ is a cocycle of local functions that represents m, and the displayed equality is an equality of functions on the nose, not simply up to coboundaries. Since g is monic, such functions are forced to be globally defined (after normalisation), and this gives the desired result; the details follow.

Fix a finite affine open cover $\mathcal{U} = \{U_i\}$ of X, and consider the cosimplicial A-algebra $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{O}_X)$ as a model for the A-algebra $\mathrm{R}\Gamma(X, \mathcal{O}_X)$. The Frobenius action $F_X^* : \mathrm{R}\Gamma(X, \mathcal{O}_X) \to \mathrm{R}\Gamma(X, \mathcal{O}_X)$

is modelled by the actual Frobenius map $X^p : x \mapsto x^p$ on each term of this A-algebra. This gives $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{O}_X)$ the structure of an $A\{X^p\}$-module, where $A\{X^p\}$ is the non-commutative polynomial ring on one generator X^p over A satisfying the commutation relation $r^p X^p = X^p r$ (see [Lau96, §1.1] for more details on this ring). In more concrete terms, at the level of cohomology, we see the following: for each polynomial $g \in A\{X^p\}$, classes $\alpha, \beta \in H^i(X, \mathcal{O}_X)$, and a scalar $r \in A$, we have $g(\alpha + \beta) = g(\alpha) + g(\beta)$, and $g(r\alpha) = r^p g(\alpha)$.

The A-finiteness of the Frobenius stable module M ensures that for any class $m \in M$, there exists a *monic* polynomial $g \in A\{X^p\}$ such that $g(m) = 0$. If we pick representatives in $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{O}_X)$ for this equation, we obtain an equation in $\mathcal{C}^i(\mathcal{U}, \mathcal{O}_X)$ of the form

$$g(\tilde{m}) = d(n)$$

where $\tilde{m} \in \mathcal{C}^i(\mathcal{U}, \mathcal{O}_X)$ is a cocycle lifting m and $n \in \mathcal{C}^{i-1}(\mathcal{U}, \mathcal{O}_X)$. As g is a monic equation, we can find a finite surjective morphism $\pi' : Y' \to X$ such that $n = g(n')$ for some $n' \in \mathcal{C}^i(\mathcal{U} \times_X Y', \mathcal{O}_{Y'})$. For example, we could do the following: for each component n_j of n (where j is a multi-index), the scheme $V_j = \mathrm{Spec}(\mathcal{O}(U_j)[T]/(g(T) - n_j))$ is a quasi-finite X-scheme such that the equation $g(n') = n$ admits a solution in $H^0(V_j, \mathcal{O}_{V_j})$. Using Proposition 5.2.1, we find a Y' and n' with the desired properties. The additivity of Frobenius now tells us that we obtain an equation in $\mathcal{C}^i(\mathcal{U} \times_X Y', \mathcal{O}_{Y'})$ of the form

$$g(\tilde{m} - d(n')) = 0.$$

The monicity of g implies that the components of $\tilde{m} - d(n')$ are integral over A. Setting Y to be an irreducible component of $Y' \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A[T]/(g(T)))$ that dominates Y' under the natural map, we find a finite surjective morphism $Y \to Y'$. The pullback of $\tilde{m} - d(n')$ in $\mathcal{C}^i(\mathcal{U} \times_X Y, \mathcal{O}_Y)$ is a vector of local functions whose components satisfy the monic polynomial g over A. As Y is integral and $H^0(Y, \mathcal{O}_Y)$ already contains roots of g, it follows that these functions are globally defined. Thus, they lie in the image of the natural map $H^0(Y, \mathcal{O}_Y) \to \mathcal{C}^\bullet(\mathcal{U} \times_X Y, \mathcal{O}_Y)$ where $H^0(Y, \mathcal{O}_Y)$ is viewed as a constant cosimplicial algebra. As the complex underlying the former cosimplicial algebra has cohomology only in degree 0, it follows that $\tilde{m} - d(n')$ is a coboundary, which implies that $\tilde{m}$ is a coboundary on Y, which shows that Y satisfies the required conditions. □

Remark 5.2.3. It would be interesting to know if Proposition 5.2.2 can be proved purely in the setting of cosimplicial $\mathbf{F}_p$-algebras. Part of the problem is that we do not know a good definition of a finite surjective morphism in the dual category of such algebras.

As a corollary of Proposition 5.2.2 and the finiteness properties enjoyed by proper morphisms, we arrive at the following result:

Corollary 5.2.4. *Let $f : X \to S$ be a proper morphism of $\mathbf{F}_p$-schemes, with S noetherian and affine. Then there exists a finite surjective morphism $\pi : Y \to X$ such that $\pi^* : H^i(X, \mathcal{O}) \to H^i(Y, \mathcal{O})$ is 0 for $i > 0$.*

Proof. The properness of X over an affine implies that $H^i(X, \mathcal{O})$ is a finite $H^0(X, \mathcal{O}_X)$-module and that $H^i(X, \mathcal{O}_X) = 0$ for i sufficiently large (see [Gro61, Corollaire 3.2.3]). Proposition 5.2.2 then finishes the proof. □

We will now finish the proof of Theorem 5.0.1. To pass from the conclusion of Corollary 5.2.4 to the general statement of Theorem 5.0.1, the obvious strategy is to cover S with affines, construct solutions that work over the affines, and take the normalisation of X in the fibre product of all of these. When carried out, this process produces a finite cover $\pi : Y \to X$ such that, with $g = f \circ \pi$, the maps $R^i f_* \mathcal{O}_X \to R^i g_* \mathcal{O}_Y$ are 0 for $i > 0$. This is not quite enough to prove the theorem: a map in $\mathrm{D}(\mathrm{Coh}(S))$ that induces the 0 map on cohomology sheaves is not necessarily zero. However, with the boundedness conditions enforced by properness, a sufficiently high iteration of this process turns out to be enough.

Proof of Theorem 5.0.1. Fix a finite affine covering $\mathcal{U} = \{U_i\}$ of S, and denote $X \times_S U_i$ by X_i. Using Corollary 5.2.4, we can find finite surjective maps $\phi_i : Z_i \to X_i$ such that the induced map $H^j(X_i, \mathcal{O}_{X_i}) \to H^j(Z_i, \mathcal{O}_{Z_i})$ is 0 for each $j > 0$. Using Proposition 5.2.1, we may find a finite surjective morphism $\phi : Z \to X$ such that ϕ_{U_i} factors through ϕ_i. This implies that $\mathrm{R}^j f_* \mathcal{O}_X \to \mathrm{R}^j (f \circ \phi)_* \mathcal{O}_Z$ is 0 for each j (as vanishing is a local statement on S). Iterating this construction $\dim(X)$ times and using Lemma 5.1.2, we obtain a proper S-scheme $g : Y \to S$ and a finite surjective S-morphism $\pi : Y \to X$ such the natural pullback map $\pi^* : \tau_{\geq 1} \mathrm{R} f_* \mathcal{O}_X \to \tau_{\geq 1} \mathrm{R} g_* \mathcal{O}_Y$ is 0, thereby proving the theorem. □

Finally, having proven Theorem 5.0.1, we point out how Theorem 5.0.2 follows. The argument given is identical to the one in Theorem 4.2.1 and only repeated here for the reader's convenience.

Proof of Theorem 5.0.2. It is clear that if S satisfies Condition 1.0.1 then it satisfies Condition 1.0.2. Thus, we equip ourselves with a scheme S satisfying Condition 1.0.2, and a proper surjective morphism $f : X \to S$. By Theorem 5.0.1, there exists a finite surjective morphism $\pi : Y \to X$ such that, with $g = f \circ \pi$, the pullback map $\tau_{\geq 1} \mathrm{R} f_* \mathcal{O}_X \to \tau_{\geq 1} \mathrm{R} g_* \mathcal{O}_Y$ is 0. By applying $\mathrm{Hom}(\mathrm{R} f_* \mathcal{O}_X, -)$ to the exact triangle

$$g_* \mathcal{O}_Y \to \mathrm{R} g_* \mathcal{O}_Y \to \tau_{\geq 1} \mathrm{R} g_* \mathcal{O}_Y \to g_* \mathcal{O}_Y[1]$$

we see that the natural pullback map $\mathrm{R} f_* \mathcal{O}_X \to \mathrm{R} g_* \mathcal{O}_Y$ factors through $g_* \mathcal{O}_Y \to \mathrm{R} g_* \mathcal{O}_Y$. As $g : Y \to S$ is a proper surjective morphism, the algebra $g_* \mathcal{O}_Y$ is a coherent sheaf of algebras corresponding to the structure sheaf of a finite surjective morphism. By assumption, the natural map $\mathcal{O}_S \to g_* \mathcal{O}_Y$ has a splitting, and thus the same is true for $\mathcal{O}_S \to \mathrm{R} f_* \mathcal{O}_X$. □

5.3 Commentary

We make a few comments about Theorem 5.0.1. In §5.3.1 we point out how to prove a version of Theorem 5.0.1 with coefficients. In §5.3.2, we point out how some assumptions in Theorem 5.0.1 cannot be dropped.

5.3.1 Some refinements

Roughly speaking, Theorem 5.0.1 says that proper morphisms behave like finite morphisms after passage to finite covers when one is working with theorems concerning the annihilation of coherent sheaf cohomology. In the following proposition, we formalise this intuition, extract a kind of "converse" to this statement, and work with non-trivial coefficients. These results will be useful in the sequel when we prove vanishing results.

Proposition 5.3.1. *Let S be a noetherian $\mathbf{F}_p$-scheme, and $f : X \to S$ be a proper surjective morphism. Then we can find a diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ {\scriptstyle a}\downarrow & \searrow^{g} & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{h} & S \end{array}$$

with π and h finite surjective morphisms such that for every locally free sheaf $\mathcal{M}$ on S and every $i \geq 0$, we have:

1. *The morphism $h^* : H^i(S, \mathcal{M}) \to H^i(S', h^*\mathcal{M})$ factors through $f^* : H^i(S, \mathcal{M}) \to H^i(X, f^*\mathcal{M})$.*
2. *The morphism $\pi^* : H^i(X, f^*\mathcal{M}) \to H^i(Y, g^*\mathcal{M})$ factors through $a^* : H^i(S', h^*\mathcal{M}) \to H^i(Y, g^*\mathcal{M})$.*

Proof. Theorem 5.0.1 gives a finite surjective morphism $\pi : Y \to X$ such that, with $g = f \circ \pi$, we have a map s and the following diagram:

$$\begin{array}{ccc} \mathcal{O}_S & \longrightarrow & g_*\mathcal{O}_Y \\ \downarrow & \nearrow^{s} & \downarrow \\ \mathrm{R}f_*\mathcal{O}_X & \longrightarrow & \mathrm{R}g_*\mathcal{O}_Y \end{array}$$

We claim that this is a commutative diagram. The triangle based at $\mathrm{R}g_*\mathcal{O}_Y$ commutes by construction. Given this commutativity, to see that the triangle based at $\mathcal{O}_S$ commutes, it suffices to show that $\mathrm{Hom}(\mathcal{O}_S, g_*\mathcal{O}_Y) \to \mathrm{Hom}(\mathcal{O}_S, \mathrm{R}g_*\mathcal{O}_Y)$ is injective. This injectivity (and, in fact, bijectivity) follows from Lemma 5.1.4. Thus, the preceding diagram is a commutative diagram in $\mathrm{D}(\mathrm{Coh}(S))$. Applying $- \otimes \mathcal{M}$, setting S' to be the Stein factorisation of $Y \to S$, and using the projection formula now gives the desired result. □

5.3.2 Possible generalisations

We have not strived to find the most general setting for Theorem 5.0.1. For example, one can easily extend the theorem to algebraic spaces or even Deligne-Mumford stacks. On the other hand, the properness hypothesis seems essential as the example below shows. In fact, the method of the proof shows that the essential property we use is that the relative cohomology classes of the structure sheaf for $f : X \to S$ are annihilated by a monic polynomial in Frobenius. We do not know if there is a better characterisation of this class of maps.

Example 5.3.2. Fix a base field k. Let $X = \mathbf{A}^2$, and $U = \mathbf{A}^2 - \{0\}$. The quotient map $U \to U/\mathbf{G}_m = \mathbf{P}^1$ gives a natural identification $H^1(U, \mathcal{O}) = \oplus_{i \in \mathbf{Z}} H^1(\mathbf{P}^1, \mathcal{O}(i))$. We claim that the non-zero classes in this group cannot be killed by a finite cover of U. To see this, note that one may view $H^1(U, \mathcal{O})$ as the local cohomology group $H^2_{\{0\}}(X, \mathcal{O}) = H^2_{\mathfrak{m}}(R)$, where $R = k[x, y]$ is the coordinate ring of X and $\mathfrak{m} = (x, y)$ is the maximal ideal corresponding to the origin. Given a finite surjective morphism $\pi : Y \to U$, we may normalise X in π to obtain a finite surjective morphism $\overline{\pi} : \overline{Y} \to X$ which contains π as the fibre over U. As before, the cohomology group $H^1(Y, \mathcal{O})$ can be viewed as $H^2_{\overline{Y} \setminus Y}(\overline{Y}, \mathcal{O})$ which, in turn, may be viewed as $H^2_{\mathfrak{m}}(S)$, where S is the coordinate ring of $\overline{Y}$ considered as an R-module in the natural way. Under these identifications, the pullback map $H^1(U, \mathcal{O}) \to H^1(Y, \mathcal{O})$ corresponds to the morphism $H^2_{\mathfrak{m}}(R) \to H^2_{\mathfrak{m}}(S)$ induced by the inclusion

$R \to S$ coming from $\overline{\pi}$. By the validity of the direct summand conjecture (proven by Hochster in [Hoc73]) for equicharacteristic regular rings such as R, the inclusion $R \to S$ is a direct summand as an R-module map. In particular, the map $H^2_{\mathfrak{m}}(R) \to H^2_{\mathfrak{m}}(S)$ is injective, which shows that the non-zero classes in $H^1(U, \mathcal{O})$ persist after passage to finite covers.

5.4 Application: A result in commutative algebra

We discuss some applications of Proposition 5.2.2 to commutative algebra. Most of these applications are implicit in [HL07]. The first result we want to dicuss is an analogue of Proposition 5.2.2 for local cohomology.

Proposition 5.4.1. *Let $(R, \mathfrak{m})$ be an excellent local noetherian $\mathbf{F}_p$-algebra such that R is finite over some ring A. For any A-finite Frobenius-stable submodule $M \in H^i_{\mathfrak{m}}(R)$ with $i \geq 1$, there exists a finite surjective morphism $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ such that $f^*(M) = 0$.*

Proof. Since R is excellent, we may pass to the normalisation and assume that R is normal. In particular, $H^i_{\mathfrak{m}}(R) = 0$ for $i = 0, 1$. For $i > 1$, we have an Frobenius equivariant identification $\delta : H^{i-1}(U, \mathcal{O}_U) \simeq H^i_{\mathfrak{m}}(R)$, where $U = \mathrm{Spec}(R) - \{\mathfrak{m}\}$ is the punctured spectrum of R. Since $i > 1$, Proposition 5.2.2 gives us a finite surjective morphism $f : V \to U$ such that $f^*(\delta^{-1}(M)) = 0$. Setting S to be the normalisation of R in V is then easily seen to do job. $\square$

Next, we dualise the Proposition 5.4.1 to obtain a global result in terms of dualising sheaves.

Proposition 5.4.2. *Let X be an excellent noetherian $\mathbf{F}_p$-scheme of equidimension d that admits a dualising complex $\omega^\bullet_X$. Then there exists a finite surjective morphism $\pi : Y \to X$ such that $\tau_{>-d}(\mathrm{Tr}_\pi) = 0$ for $i > 0$, where Tr_π is the trace map $\mathrm{Tr}_\pi : \pi_*\omega^\bullet_Y \to \omega^\bullet_X$.*

Proof. Fix an integer $i > 0$. We prove the claim by induction on the dimension $d = \dim(X)$. We may assume that $d > 0$ as the there is nothing to prove when the dimension is 0. By repeated iterations of Lemma 5.1.2, it suffices to find a finite surjective morphism $\pi : Y \to X$ such that $\mathcal{H}^{-d+i}(\mathrm{Tr}_\pi) = 0$. As vanishing of a map of sheaves is a local statement, we reduce to the case that X is an excellent noetherian local $\mathbf{F}_p$-algebra $(R, \mathfrak{m})$ admitting a dualising complex. For each non-maximal $\mathfrak{p} \in \mathrm{Spec}(R)$, we can inductively find a finite morphism $\pi_{\mathfrak{p}} : Y_{\mathfrak{p}} \to \mathrm{Spec}(R_{\mathfrak{p}})$ such that $\mathcal{H}^{-d_{R_{\mathfrak{p}}}+i}(\mathrm{Tr}_{\pi_{\mathfrak{p}}})$ is the 0 map. As explained in Chapter 2, the R-module $\mathcal{H}^{-d+i}(\omega^\bullet_R)$ localises to $\mathcal{H}^{-d_{R_{\mathfrak{p}}}+i}(\omega^\bullet_{R_{\mathfrak{p}}})$ at $\mathfrak{p}$. Hence, the normalisation $\overline{\pi_{\mathfrak{p}}} : \overline{Y_{\mathfrak{p}}} \to X$ induces the 0 map on $\mathcal{H}^{-d+i}(\mathrm{Tr}_\pi)$ when localised at $\mathfrak{p}$. Finding such a cover for each non-maximal prime $\mathfrak{p}$ in the finite set of associated primes of $\mathcal{H}^{-d+i}(\omega^\bullet_R)$ and normalising X in the fibre product of the resulting collection, we find a cover $\pi : Y \to X$ such that $\mathcal{H}^{-d+i}(\mathrm{Tr}_\pi)$ has an image supported only at the closed point. Setting $Y = \mathrm{Spec}(S)$, duality tells us that the image M of $H^{d-i}_{\mathfrak{m}}(R) \to H^{d-i}_{\mathfrak{m}}(S)$ is a finite length Frobenius-stable R-submodule. Proposition 5.4.1 then allows us to find a finite surjective morphism $g : \mathrm{Spec}(T) \to \mathrm{Spec}(S)$ such that $g^*(M) = 0$. It follows that the composite map $\pi' : \mathrm{Spec}(T) \to \mathrm{Spec}(R)$ induces the 0 map on $H^{d-i}_{\mathfrak{m}}(R)$. By duality, we see that $\mathcal{H}^{-d+i}(\mathrm{Tr}_{\pi'}) = 0$ as desired. $\square$

Using Proposition 5.4.1, we discover that rings satisfying Condition 1.0.2 are Cohen-Macaulay. We will use this result in Chapter 7 when comparing Condition 1.0.2 to F-rationality.

Corollary 5.4.3. *Let $(R, \mathfrak{m})$ is an excellent noetherian local $\mathbf{F}_p$-algebra satisfying Condition 1.0.2. Assume that R admits a dualising complex. Then R is a normal Cohen-Macaulay domain.*

Proof. The normality of R follows from Remark 3.1.6. To verify that R is Cohen-Macaulay, it suffices to show that $\omega_R^\bullet$ is concenctrated in degree d where $d = \dim(R)$, i.e., that $\mathcal{H}^{-d+k}(\omega_R^\bullet) = 0$ for $k > 0$. By Proposition 5.4.2, we can find a finite surjective morphism $\pi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ such that $\mathcal{H}^{-d+k}(\mathrm{Tr}_\pi) = 0$, where $\mathrm{Tr}_\pi : \pi_*\omega_S^\bullet \to \omega_R^\bullet$. Since R satisfies Condition 1.0.2, the inclusion $R \to S$ is a direct summand. Applying $\mathrm{R}\mathcal{H}\mathrm{om}(-, \omega_R^\bullet)$, we see that the trace map Tr_π is the projection onto a summand. Hence, the assumption that $\mathcal{H}^{-d+k}(\mathrm{Tr}_\pi) = 0$ implies that $\mathcal{H}^{-d+k}(\omega_R^\bullet) = 0$, as desired. □

Remark 5.4.4. One key ingredient in the proof of Proposition 5.4.2 is the good behaviour of local cohomology and dualising sheaves with respect to localisation. This behaviour, already mentioned in Chapter 2, was first observed by Grothendieck in [Gro68a, Exposé VIII, Théorème 2.1] where it is used to show the following: a noetherian local ring $(R, \mathfrak{m})$ of dimension d that is Cohen-Macaulay outside the closed point and admits a dualising complex has the property that $H^i_\mathfrak{m}(R)$ has finite length for $i < d$. This argument can also be found in the main theorem [HL07].

5.5 Application: A question of Karen Smith

The main result of Hochster-Huneke [HH92] is a result in commutative algebra. While geometrising it in [Smi97c], K. Smith arrived at the following question (see [Smi97a]):

Question 5.5.1. Let X be a projective variety over a field k of characteristic p, and let $\mathcal{L}$ be a "weakly positive" line bundle on X. For any $n \in \mathbf{Z}$ and any $0 < i < \dim(X)$, does there exist a finite surjective morphism $\pi : Y \to X$ such that $H^i(X, \mathcal{L}^{\otimes n}) \to H^i(Y, \pi^*\mathcal{L}^{\otimes n})$ is 0?

Using the algebraic result of Hochster-Huneke [HH92], one can show that if we take "weakly positive" to mean ample, then Question 5.5.1 has an affirmative answer (see Remark 5.5.4). Smith had originally hoped that "weakly positive" could be taken to mean nef. We give some examples in the sequel to show that this cannot be the case. However, first, we prove some positive results.

5.5.1 Positive results

We first examine Question 5.5.1 in the case of positive twists. It is clear that being ample is a sufficiently positive condition for the required vanishing statement to be true: Frobenius twisting can be realised by pulling back along a finite morphism and has the effect of changing $\mathcal{L}$ by $\mathcal{L}^{\otimes p}$, whence Serre vanishing shows the desired result. It is natural to wonder if the result passes to the closure of the ample cone, i.e., the nef cone. We show in Example 5.5.8 that this is not the case: there exist non-torsion degree 0 line bundles on surfaces whose middle cohomology cannot be killed by finite covers. On the other hand, Corollary 5.2.4 coupled with the fact that torsion line bundles can be replaced with $\mathcal{O}$ on passage to a finite cover ensures that Question 5.5.1 has a positve answer for torsion line bundles. The necessity of the non-torsion requirement and the observation that torsion line bundles are semiample suggested that the following Proposition might be true.

Proposition 5.5.2. *Let X be a proper variety over a field of characteristic p, and let $\mathcal{L}$ be a semi-ample line bundle on X. For any $i > 0$, there exists a finite surjective morphism $\pi : Y \to X$ such that the induced map $H^i(X, \mathcal{L}) \to H^i(Y, \pi^*\mathcal{L})$ is* 0.

Proof. As $\mathcal{L}$ is a semiample bundle, there exists some positive integer m such that $\mathcal{L}^{\otimes m}$ is globally generated. If we fix a basis $s_1, \ldots, s_k$ for $H^0(X, \mathcal{L}^{\otimes m})$, then the cyclic covering trick (see [Laz04a, Proposition 4.1.3]) ensures that there's a finite flat cover $\pi : \tilde{X} \to X$ such that $\pi^*(s_i)$ admits an m-th root in $H^0(\tilde{X}, \pi^*\mathcal{L})$ and, consequently, $\pi^*\mathcal{L}$ is globally generated. In particular, as semiamplitude is preserved under pullbacks, we may replace X with $\tilde{X}$ and assume that $\mathcal{L}$ arises as the pullback of an ample bundle $\mathcal{M}$ under a proper surjective morphism $f : X \to S$. Furthermore, once $f : X \to S$ is fixed, to show the required vanishing statement, we may always replace $\mathcal{L}$ by $\mathcal{L}^{\otimes p^j}$ for $j \gg 0$ because the Frobenius morphism $F_X : X \to X$ is finite surjective with $F_X^*\mathcal{L} = \mathcal{L}^{\otimes p}$. Now the projection formula for f implies that $\mathrm{R}f_*(\mathcal{L}^{\otimes p^j}) = \mathrm{R}f_*\mathcal{O}_X \otimes_S^{\mathrm{L}} \mathcal{M}^{\otimes p^j}$. Using Theorem 5.0.1, we may find a finite surjective morphism $\pi : Y \to X$ such that, with $g = f \circ \pi$, we have a factorisation $\mathrm{R}f_*(\mathcal{L}^{\otimes p^j}) \to g_* f^*(\mathcal{L}^{\otimes p^j}) \to \mathrm{R}g_*\pi^*(\mathcal{L}^{\otimes p^j})$ of the natural map $\pi^* : \mathrm{R}f_*(\mathcal{L}^{\otimes p^j}) \to \mathrm{R}g_*\pi^*(\mathcal{L}^{\otimes p^j})$. Applying $H^i(S, -)$ to the composite morphism gives us the desired morphism. Thus, to show the required statement, it suffices to show that $H^i(S, g_*\pi^*(\mathcal{L}^{\otimes p^j})) = 0$ for $j \gg 0$. By the projection formula, we have

$$H^i(S, g_*\pi^*(\mathcal{L}^{\otimes p^j})) = H^i(S, g_*g^*(\mathcal{M}^{\otimes p^j})) = H^i(S, g_*\mathcal{O}_Y \otimes \mathcal{M}^{\otimes p^j})$$

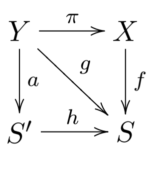
As $\mathcal{M}$ is ample, this group vanishes by Serre vanishing for $j \gg 0$, as required. □

Based on Proposition 5.5.2, one might expect that semiamplitude is a positive enough property for Question 5.5.1 to have an affirmative answer for the case of negative twists as well. We show in Example 5.5.9 that this is not the case; the key feature of that example is that the semiample line bundle defines a map that's not generically finite. In fact, this feature is essentially the only obstruction: if $\mathcal{L}$ is both semiample and big, then Question 5.5.1 has an affirmative answer even for negative twists of $\mathcal{L}$.

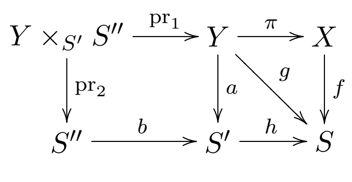
Proposition 5.5.3. *Let X be a proper variety over a field of characteristic p, and let $\mathcal{L}$ be a semi-ample and big line bundle on X. For any $i < \dim(X)$, we can find a finite surjective morphism $\pi : Y \to X$ such that the induced map $H^i(X, \mathcal{L}^{-1}) \to H^i(Y, \pi^*\mathcal{L}^{-1})$ is 0*

Proof. We first describe the idea informally. Using Proposition 5.3.1 and arguments similar to those in the proof of Proposition 5.5.2, we will reduce to the case that $\mathcal{L}$ is actually *ample* on X. In this case, we give a direct proof using Proposition 5.4.2; the details follow.

Fix an integer $i < \dim(X)$. As $\mathcal{L}$ is big, there is nothing to show for $i = 0$ and, thus, we may assume $i > 0$. As in the proof of Proposition 5.5.2, at the expense of replacing X by a finite flat cover, we may assume that $\mathcal{L}$ arises as the pullback of an ample line bundle $\mathcal{M}$ under a proper surjective morphism $f : X \to S$. As bigness is preserved under passage to finite flat covers, we may continue to assume that $\mathcal{L}$ is big. In particular, the map f is forced to be an alteration. By Proposition 5.3.1, we can find a diagram



with π and h finite surjective, such that we have a factorisation $H^i(X, \mathcal{L}^{-1}) \xrightarrow{s} H^i(S', h^*\mathcal{M}^{-1}) \xrightarrow{a^*} H^i(Y, \pi^*\mathcal{L}^{-1})$ of π^* for some map s. Moreover, given a finite cover $b : S'' \to S$, we can form the diagram



This means that at the level of cohomology, we have a commutative diagram

$$\begin{array}{ccc} H^i(X, \mathcal{L}^{-1}) & \xrightarrow{(\pi \circ \mathrm{pr}_1)^*} & H^i(Y \times_{S'} S'', (\pi \circ \mathrm{pr}_1)^* \mathcal{L}^{-1}) \\ \downarrow{\scriptstyle s} & & \uparrow{\scriptstyle \mathrm{pr}_2^*} \\ H^i(S', h^* \mathcal{M}^{-1}) & \xrightarrow{b^*} & H^i(S'', b^* h^* \mathcal{M}^{-1}) \end{array}$$

Thus, it suffices to show that $H^i(S', h^*\mathcal{M}^{-1})$ can be killed by finite covers of S'. As h is a finite morphism, the bundle $h^*\mathcal{M}$ is ample. That f was an alteration forces $\dim(S') = \dim(X)$ and, therefore, $0 < i < \dim(S')$. In other words, we are reduced to verifying the claim in the theorem under the additional assumption that $\mathcal{L}$ is ample.

As we are free to replace X by a Frobenius twist (which increases the positivity of $\mathcal{L}$), we may assume that $\mathcal{L}$ has the property that $H^j(X, \mathcal{L} \otimes \omega_X) = 0$ for all $j > 0$, where ω_X is the dualising *sheaf* on X. Now choose a finite surjective morphism $\pi : Y \to X$ satisfying the conclusion of Proposition 5.4.2. With $d = \dim(X)$, the trace map induces the following morphism of triangles in $\mathrm{D}^b(\mathrm{Coh}(X))$:

$$\begin{array}{ccccccc} \pi_*\omega_Y[d] & \longrightarrow & \pi_*\omega_Y^\bullet & \longrightarrow & \tau_{>-d}\pi_*\omega_Y^\bullet & \longrightarrow & \pi_*\omega_Y[d+1] \\ \downarrow{\scriptstyle a} & \swarrow{\scriptstyle s} & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c=0} & & \downarrow \\ \omega_X[d] & \longrightarrow & \omega_X^\bullet & \longrightarrow & \tau_{>-d}\omega_X^\bullet & \longrightarrow & \omega_X[d+1]. \end{array}$$

Here s is a map that whose existence is ensured by the equation $c = 0$ (but s is not necessarily unique). Tensoring this diagram with $\mathcal{L}$, using the flatness of $\mathcal{L}$, and using the projection formula gives us the following morphism of triangles:

$$\begin{array}{ccccccc} \pi_*(\omega_Y \otimes \pi^*\mathcal{L})[d] & \longrightarrow & \pi_*(\omega_Y^\bullet \otimes \pi^*\mathcal{L}) & \longrightarrow & \tau_{>-d}\pi_*(\omega_Y^\bullet \otimes \pi^*\mathcal{L}) & \longrightarrow & \pi_*(\omega_Y \otimes \pi^*\mathcal{L})[d+1] \\ \downarrow{\scriptstyle a_\mathcal{L}} & \swarrow{\scriptstyle s_\mathcal{L}} & \downarrow{\scriptstyle b_\mathcal{L}} & & \downarrow{\scriptstyle c_\mathcal{L}=0} & & \downarrow \\ \omega_X \otimes \mathcal{L}[d] & \longrightarrow & \omega_X^\bullet \otimes \mathcal{L} & \longrightarrow & \tau_{>-d}\omega_X^\bullet \otimes \mathcal{L} & \longrightarrow & \omega_X \otimes \mathcal{L}[d+1]. \end{array}$$

The commutativity of the above diagram and existence of $s_\mathcal{L}$ shows that for any integer i, the image of the natural trace map

$$H^{-i}(b_\mathcal{L}) : H^{-i}(Y, \omega_Y^\bullet \otimes \pi^*\mathcal{L}) \to H^{-i}(X, \omega_X^\bullet \otimes \mathcal{L})$$

lies in the image of the natural map

$$H^{d-i}(X, \omega_X \otimes \mathcal{L}) = H^{-i}(X, \omega_X \otimes \mathcal{L}[d]) \to H^{-i}(X, \omega_X^\bullet \otimes \mathcal{L}).$$

Now choose i such that $0 < i < d$. By assumption, the source of the preceding map is then trivial. Hence, we find that the map $H^{-i}(b_\mathcal{L})$ is also 0. Dualising, it follows that

$$\pi^* : H^i(X, \mathcal{L}^{-1}) \to H^i(Y, \pi^*\mathcal{L}^{-1})$$

is trivial, as desired. □

Remark 5.5.4. Consider the special case of Proposition 5.5.3 when $\mathcal{L}$ is ample. We treated this case directly in the second half of the proof above using Proposition 5.4.2. It is possible to replace this part of the proof by a reference to [HH92, Theorem 1.2], the main geometric theorem of that paper. We have not adopted this approach as we feel that the proof given above using Proposition 5.4.2 is cleaner than the algebraic approach of [HH92] which involves developing a theory of graded integral closures (see [HH92, §4]) and reducing to a local algebra theorem.

Remark 5.5.5. Proposition 5.5.3 can be viewed as a weaker version of Kawamata-Viehweg vanishing in characteristic p up to finite covers. The natural way to approach this question is to ask for liftability to the 2-truncated Witt vector ring $\mathrm{W}_2(k)$ by smooth varieties up to finite covers, and then quote Deligne-Illusie [DI87]. In fact, thanks to Proposition 5.3.1, it would suffice to show that any variety can be dominated by a smooth one that lifts to $\mathrm{W}_2(k)$. Unfortunately, we do not know the answer to this.

5.5.2 Counterexamples

This section is dedicated to providing the examples promised earlier. Our first example is that of a degree 0 line bundle $\mathcal{M}$ on a curve C whose top cohomology cannot be killed by finite covers of C. This shows that the semiamplitude hypothesis in Proposition 5.5.2 cannot be weakened to a nefness hypothesis.

Example 5.5.6. Fix a curve C of genus $g(C) \geq 2$ over an uncountable field k of characteristic p, and let $\mathcal{M}$ be a very general degree 0 line bundle on C. We will show that $H^1(C, \mathcal{M})$ cannot be killed by finite covers. If not, by normalising if necessary, we have a finite flat map $f : C' \to C$ such that $f^* : H^1(C, \mathcal{M}) \to H^1(C', f^*\mathcal{M})$ is the 0 map. Furthermore, by replacing C' with a cover if necessary, we may even assume that the extension of function fields induced by f is normal. Tensoring the exact sequence

$$0 \to \mathcal{O}_C \to f_*\mathcal{O}_{C'} \xrightarrow{q} \mathcal{Q} \to 0$$

with $\mathcal{M}$ and using the projection formula tells us that $H^0(C, \mathcal{M} \otimes \mathcal{Q}) \neq 0$ or, equivalently, that $\mathcal{M}^{-1}$ occurs as a subsheaf of $\mathcal{Q}$. We will analyse the Harder-Narasimhan filtration of $f_*\mathcal{O}_{C'}$ to show that this cannot happen if $\mathcal{M}$ is chosen to be very general. We refer the reader to [Laz04b, §6.4.A] for generalities on the Harder-Narasimhan filtration.

A theorem of Lazarsfeld from the Appendix of [PS00] implies that $\mathcal{Q}^\vee$ is a nef vector bundle, i.e., for any subbundle $\mathcal{E} \hookrightarrow f_*\mathcal{O}_C$, the quotient $q(\mathcal{E})^\vee$ of $\mathcal{Q}^\vee$ is nef. The bundle E, thus being an extension of the antinef vector bundle $q(\mathcal{E})$ by a subsheaf of $\mathcal{O}_C$, has non-positive degree. This implies that the maximal slope occuring in the Harder-Narasimhan filtration for $f_*\mathcal{O}_{C'}$ is 0. Thus, as $\mathcal{O}_C$ is a maximal degree subbundle of $f_*\mathcal{O}_C$, we have an exact sequence of semistable degree 0 vector bundles

$$0 \to \mathcal{O}_C \to \mathrm{Fil}^0(f_*\mathcal{O}_{C'}) \to \mathrm{Fil}^0(\mathcal{Q}) \to 0.$$

As the full subcategory $\mathrm{Vect}(C)_0^{\mathrm{ss}}$ of $\mathrm{Coh}(C)$ spanned by semistable degree 0 vector bundles is an abelian category with simples corresponding to stable vector bundles, any stable vector bundle occuring as a Jordan-Hölder constituent (i.e., a simple subquotient) of $\mathrm{Fil}^0(\mathcal{Q}) \in \mathrm{Vect}(C)_0^{\mathrm{ss}}$ also occurs in $\mathrm{Fil}^0(f_*\mathcal{O}_{C'})$. In particular, the line bundle $\mathcal{M}^{-1}$ occurs in $\mathrm{Fil}^0(f_*\mathcal{O}_{C'})$. The latter vector bundle inherits an algebra structure from $f_*\mathcal{O}_{C'}$ and, therefore, corresponds to a curve $g : C'' \to C$ with $g_*\mathcal{O}_{C''} = \mathrm{Fil}^0(f_*\mathcal{O}_{C'})$. The fact that $\mathrm{Vect}(C)_0^{\mathrm{ss}}$ is artinian and noetherian as an abelian category implies that only finitely many degree 0 line bundles occur in $g_*\mathcal{O}_{C''}$. Moreover, as f was generically normal, so is g. Thus, our desired result will now follow if we show that the collection of generically normal finite flat $g : C'' \to C$ with $\deg(g_*\mathcal{O}_{C''}) = 0$ is countable.

As g is generically normal, it can be factored as $C'' \xrightarrow{h} \tilde{C} \xrightarrow{F} C$ with h generically étale, and F is purely inseparable. As any purely inseparable map is dominated by some power of the Frobenius map, there are only countably many possibilities for F. On the other hand, $\deg(g_*\mathcal{O}_{C''}) = 0$ implies that $\deg(h_*\mathcal{O}_{C''}) = 0$. By Riemann-Roch, it follows that $\chi(C'') = \deg(h)\chi(\tilde{C})$. As h is generically étale, Riemann-Hurwitz then applies to say that h is finite étale. As F is purely inseparable, it induces an isomorphism $\pi_1(F) : \pi_1(\tilde{C}) \xrightarrow{\simeq} \pi_1(C)$. In particular, by the finite generation of the étale fundamental group (proven by lifting the curve and the cover to characteristic 0, for example), there are only countably many possibilities for h. Thus, there are only countably many possiblities for the pair (F, h) and, therefore, for g, proving the claim.

Remark 5.5.7. Example 5.5.6 requires us to work with very general line bundles. Thus, it does not answer the following question: does the conclusion of Proposition 5.5.2 hold for nef line bundles provided the base field is $\overline{\mathbf{F}}_p$? We do not know the answer to this. A natural place to look for a counterexample would be the surfaces and threefolds considered in [Tot09].

Using Example 5.5.6, we can easily produce an example of a nef line bundle $\mathcal{L}$ on a surface X whose middle cohomology cannot be killed by passage to finite covers. In fact, the bundle constructed has degree 0 and can thus be viewed as the inverse of a nef bundle as well; this dual perspective negatively answers Question 5.5.1 for the case of positive or negative twists when "weakly positive" is taken to mean nef.

Example 5.5.8. Let $(C, \mathcal{M})$ be as in Example 5.5.6. Then $\mathcal{L} = \mathcal{M} \boxtimes \mathcal{O}_C = \mathrm{pr}_1^*\mathcal{M}$ is a nef line bundle on $X = C \times C$ with $\mathrm{pr}_1^* : H^1(C, \mathcal{M}) \xrightarrow{\simeq} H^1(X, \mathrm{pr}_1^*\mathcal{M}) = H^1(X, \mathcal{L})$. We claim that there does not exist a finite surjective morphism $\pi : Y \to X$ inducing the 0 map $\pi^* : H^1(X, \mathcal{L}) \to H^1(Y, \pi^*\mathcal{L})$. If π was such a map, then choosing a multisection of $\mathrm{pr}_1 \circ \pi$ and normalising it gives a finite flat morphism $f : C' \to C$ inducing the 0 map on $H^1(C, \mathcal{M})$. However, as shown in Example 5.5.6, this cannot happen.

Our next example is that of a semiample line bundle $\mathcal{L}$ on a surface X such that the middle cohomology of $\mathcal{L}^{-1}$ cannot be killed by finite covers. Thus, it negatively answers Question 5.5.1 for the case of negative twists when "weakly positive" is taken to mean even semiample, not just nef.

Example 5.5.9. Consider the bundle $\mathcal{L} = \mathcal{O}(2) \boxtimes \mathcal{O} = \mathrm{pr}_1^*\mathcal{O}(2)$ on $X = \mathbf{P}^1 \times \mathbf{P}^1$ over a field k. This is a semiample bundle with $H^1(X, \mathcal{L}^{-1}) = H^1(\mathbf{P}^1, \mathcal{O}(-2)) \otimes H^0(\mathbf{P}^1, \mathcal{O}) = k$. We claim that there is no finite surjective morphism $g : Y \to X$ inducing the 0 map $H^1(X, \mathcal{L}^{-1}) \to H^1(Y, \pi^*\mathcal{L}^{-1})$. If there were such a map g, then $\mathrm{pr}_1 \circ g : Y \to \mathbf{P}^1$ is an alteration inducing the 0 map on $H^1(\mathbf{P}^1, \mathcal{O}(-2))$. Choosing a multisection of $\mathrm{pr}_1 \circ g$ and normalising it gives a finite flat morphism $f : C \to \mathbf{P}^1$ inducing the 0 map on $H^1(\mathbf{P}^1, \mathcal{O}(-2))$. However, this cannot happen: the morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-2) & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}_0 \oplus \mathcal{O}_\infty & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & f_*f^*\mathcal{O}(-2) & \longrightarrow & f_*\mathcal{O} & \longrightarrow & f_*\mathcal{O}_{f^{-1}(0)} \oplus f_*\mathcal{O}_{f^{-1}(\infty)} & \longrightarrow & 0 \end{array}$$

gives us a morphism of exact sequences

$$\begin{array}{ccccc} k = H^0(\mathbf{P}^1, \mathcal{O}) & \hookrightarrow & H^0(\mathbf{P}^1, \mathcal{O}_0 \oplus \mathcal{O}_\infty) & \stackrel{a}{\twoheadrightarrow} & H^1(\mathbf{P}^1, \mathcal{O}(-2)) = k \\ \downarrow \simeq & & \downarrow b & & \downarrow d \\ k = H^0(C, \mathcal{O}) & \stackrel{c}{\hookrightarrow} & H^0(C, \mathcal{O}_{f^{-1}(0)} \oplus \mathcal{O}_{f^{-1}(\infty)}) & \longrightarrow & H^1(C, f^*\mathcal{O}(-2)) \end{array}$$

The surjectivity of a gives that $\dim(H^0(\mathbf{P}^1, \mathcal{O}_0 \oplus \mathcal{O}_\infty)) = 2$, while the injectivity of b ensures that $\dim(\mathrm{im}(b)) = 2$. As $\dim(\mathrm{im}(c)) = 1$, it follows that $\mathrm{im}(b)$ strictly contains $\mathrm{im}(c)$, and therefore, $\dim(\mathrm{im}(d)) = 1$ which is what we wanted.

Finally, we conclude by giving an example showing that the conclusion of Proposition 5.5.3 fails for nef and big line bundles; we invite the reader to contrast this with the situation in characteristic 0 or even the liftable case (see Remark 5.5.5). The example builds on the one in Example 5.5.6,

Example 5.5.10. Let $(C, \mathcal{M})$ be as in Example 5.5.6, and let $\mathcal{L}$ be an ample line bundle on C. Let $\mathcal{E} = \mathcal{L} \oplus \mathcal{M}$, let $X = \mathbf{P}(\mathcal{E})$, and let $\pi : X \to C$ be the natural projection. With $\mathcal{O}_\pi(1)$ denoting the Serre line bundle on X, we will show the following:

- The line bundle $\mathcal{O}_\pi(1)$ is nef.
- The line bundle $\mathcal{O}_\pi(1)$ is big.
- The group $H^1(X, \mathcal{O}_\pi(1))$ is non-zero, and cannot be annihilated by finite covers of X.

We will first verify that $\mathcal{O}_\pi(1)$ is nef. Using the Barton-Kleiman criterion (see [Laz04b, Proposition 6.1.18]), it suffices to show that for any quotient $\mathcal{E} \twoheadrightarrow \mathcal{N}$ with $\mathcal{N}$ invertible, we must have $\deg(\mathcal{N}) \geq 0$. This claim follows from the formula

$$\mathrm{Hom}(\mathcal{E}, \mathcal{N}) = \mathrm{Hom}(\mathcal{L}, \mathcal{N}) \oplus \mathrm{Hom}(\mathcal{M}, \mathcal{N})$$

and the fact that neither line bundle $\mathcal{L}$ nor $\mathcal{M}$ admits a map to a line bundle with negative degree.

We now verify bigness of $\mathcal{O}_\pi(1)$. By definition, this amounts to showing that $h^0(X, \mathcal{O}_\pi(n))$ grows quadratically in n (we follow the usual convention that $h^0(X, \mathcal{F}) = \dim(H^0(X, \mathcal{F}))$ for a coherent sheaf $\mathcal{F}$ on X). Standard calculations about projective space bundles show that

$$\pi_* \mathcal{O}_\pi(n) \simeq \mathrm{R}\pi_* \mathcal{O}_\pi(n) \simeq \mathrm{Sym}^n(\mathcal{E})$$

for $n > 0$. The Leray spectral sequence for π then gives us that

$$H^0(X, \mathcal{O}_\pi(n)) = H^0(C, \mathrm{Sym}^n(E)) = H^0(C, \oplus_{i+j=n} \mathcal{L}^i \otimes \mathcal{M}^j).$$

Since $\mathcal{L}$ is ample and $\mathcal{M}$ has degree 0, the Riemann-Roch estimate tells us that $H^0(C, \mathcal{L}^i \otimes \mathcal{M}^j)$ grows like i (for big enough i). Hence, we find

$$h^0(X, \mathcal{O}_\pi(n)) = \sum_{i+j=n} h^0(C, \mathcal{L}^i \otimes \mathcal{M}^j) \sim 1 + 2 + \cdots + n = \frac{n(n-1)}{2},$$

thereby verifying the bigness of $\mathcal{O}(1)$.

To show the last claim, note that the Leray spectral sequence also shows that

$$H^1(X, \mathcal{O}_\pi(1)) = H^1(C, \mathcal{E}) = H^1(C, \mathcal{L}) \oplus H^1(C, \mathcal{M}).$$

In particular, this group is non-zero since the second factor is so. Moreover, the natural projection $\mathcal{E} \twoheadrightarrow \mathcal{M}$ defines a section $s : C \to X$ of π such that $s^* \mathcal{O}_\pi(1) \simeq \mathcal{M}$. Hence, we find that s^* induces a map $H^1(X, \mathcal{O}_\pi(1)) \to H^1(C, \mathcal{M})$ which is simply the projection on the second factor under the preceding isomorphism. In particular, if there was a finite cover $\pi : Y \to X$ such that $\pi^*(H^1(X, \mathcal{O}_\pi(1)) = 0$, then restricting Y to $s : C \to X$, we would obtain a finite cover of C annihilating $H^1(C, \mathcal{M})$, contradicting what we proved in Example 5.5.6.

5.6 Application: Some more global examples

In §4.2.3, we discussed a few examples of projective varieties satisfying Condition 1.0.2. The key feature of all those examples was that they were toric varieties. Our goal in this section is to add a non-toric example to this list: the complete Flag variety.

We first record an elementary criterion to test when a finite morphism is "split" in the sense of Condition 1.0.2.

Lemma 5.6.1. *Let X be a Gorenstein projective scheme of equidimension n over a field k, and let $\pi : Y \to X$ be a proper morphism. Then the existence of a section of $\mathcal{O}_X \to \mathrm{R}\pi_*\mathcal{O}_Y$ is equivalent to the injectivity of $H^n(X, \omega_X) \to H^n(Y, \pi^*\omega_X)$*

Proof. By the projection formula and the flatness of ω_X, we have $H^n(Y, \pi^*\omega_X) = H^n(X, \omega_X \otimes \mathrm{R}\pi_*\mathcal{O}_Y)$. Thus, the injectivity of $H^n(X, \omega_X) \to H^n(Y, \pi^*\omega_X)$ is equivalent to the injectivity of

$$H^n(X, \omega_X) \to H^n(X, \omega_X \otimes \mathrm{R}\pi_*\mathcal{O}_Y).$$

This map is the map on H^n induced by the natural map $\omega_X \to \omega_X \otimes \mathrm{R}\pi_*\mathcal{O}_Y$. Serre duality (see Chapter 2) tells us that this injectivity is equivalent to the surjectivity of

$$\mathrm{Hom}(\mathrm{R}\pi_*\mathcal{O}_Y \otimes \omega_X, \omega_X) \to \mathrm{Hom}(\omega_X, \omega_X).$$

Since ω_X is invertible, the preceding surjectivity is equivalent to the surjectivity of

$$(\pi^*)^\vee = \mathrm{ev}_1 : \mathrm{Hom}(\mathrm{R}\pi_*\mathcal{O}_Y, \mathcal{O}_X) \to \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X)$$

induced by the natural map $\mathcal{O}_X \to \mathrm{R}\pi_*\mathcal{O}_Y$. On the other hand, the surjectivity of this map is also clearly equivalent to $\mathcal{O}_X \to \mathrm{R}\pi_*\mathcal{O}_Y$ admitting a section; the claim follows. □

We now record a criterion that allows us to pass from subvarieties satisfying Condition 1.0.2 to the entire variety. The criterion is formulated in terms the existence of nice resolutions of dualising sheaves.

Proposition 5.6.2. *Let X be a Gorenstein projective variety of equidimension n over a field k of positive characteristic p. Let $i : Z \hookrightarrow X$ be a closed equidimensional subvariety that is itself Gorenstein, and let c be the codimension $\dim(X) - \dim(Z)$. Assume that there exists a resolution of ω_Z of the following form:*

$$[\omega_X = \mathcal{E}_c \to \mathcal{E}_{c-1} \to \cdots \to \mathcal{E}_0] \simeq \omega_Z$$

where, for each $0 \leq i < c$, the sheaf $\mathcal{E}_i$ is an iterated extension of inverses of semiample and big line bundles. If Z satisfies Condition 1.0.2, then so does X.

Proof. We will verify Condition 1.0.1. Let $\alpha \in H^{n-c}(X, \omega_Z) \simeq H^{n-c}(Z, \omega_Z)$ be a generator (under Serre duality). Let $f : Y \to X$ be an alteration. By assumption on Z, we know that $f^*(\alpha)$ is not zero in $H^{n-c}(Y, f^*\omega_Z)$. Given the natural map $\mathrm{L}f^*\omega_Z \to f^*\omega_Z$, we find that the pullback $\mathrm{L}f^*\alpha \in H^{n-c}(Y, \mathrm{L}f^*\omega_Z)$ is also non-zero. Note that this holds for *any* alteration $f : Y \to X$; this observation will be applied later in the proof to a different map.

Pulling back the given resolution for ω_Z to Y, we obtain a resolution

$$[f^*\omega_X = f^*\mathcal{E}_c \to f^*\mathcal{E}_{c-1} \to \cdots \to f^*\mathcal{E}_0] \simeq \mathrm{L}f^*\omega_Z$$

The hypercohomology spectral sequence associated to the stupid filtration of this complex takes the form:

$$E_p^{1,q}(Y \to X) : H^q(Y, f^*\mathcal{E}_p) \Rightarrow H^{q-p}(Y, \mathrm{L}f^*\omega_Z)$$

We will trace the behaviour of the class $\mathrm{L}f^*\alpha \in H^{n-c}(Y, \mathrm{L}f^*\omega_Z)$ through the spectral sequence. The terms contributing to this group in the spectral sequence are $H^q(Y, f^*\mathcal{E}_p)$ with $q - p = n - c$. Since $\dim(Y) = n$, the contributing terms $H^q(Y, f^*\mathcal{E}_p)$ have $q < n$ whenever $p < c$. We will first show by applying Proposition 5.5.3 that these numerics imply that $\mathrm{L}f^*\alpha$ has to be non-zero in $H^n(Y, f^*\mathcal{E}_0)$, and then we will explain why this is enough to prove the claim.

Since the bundles $\mathcal{E}_i$ are assumed to be iterated extensions of inverses of semiample and big line bundles for $i < c$, the same is true for the pullbacks $f^*\mathcal{E}_i$. Proposition 5.5.3 then allows us to produce a finite surjective morphism $g : Y' \to Y$ such that $H^j(Y, f^*\mathcal{E}_i) \to H^j(Y', g^*f^*\mathcal{E}_i)$ is 0 for $j < n$ (and $i < c$ still). Since we know that $\mathrm{L}(f \circ g)^*\alpha$ is non-zero by the earlier argument, it follows that the image of $\mathrm{L}f^*\alpha$ has to be non-zero in $H^n(Y, f^*\mathcal{E}_0) = H^n(Y, f^*\omega_X)$ under the natural coboundary map $H^{n-c}(Y, \mathrm{L}f^*\omega_Z) \to H^n(Y, f^*\omega_X)$.

Now note that we also have a analogous spectral sequence

$$E_p^{1,q}(X \to X) : H^q(X, \mathcal{E}_p) \Rightarrow H^{q-p}(X, \omega_Z)$$

and a morphism of spectral sequences $E_p^{1,q}(X \to X) \to E_p^{1,q}(Y \to X)$ by pulling back classes. This gives rise to the commutative square

$$\begin{array}{ccc} k \simeq H^{n-c}(X, \omega_Z) & \xrightarrow{\delta_X} & H^n(X, \omega_X) \simeq k \\ \downarrow a & & \downarrow b \\ H^{n-c}(Y, \mathrm{L}f^*\omega_Z) & \xrightarrow{\delta_Y} & H^n(Y, f^*\omega_X) \end{array}$$

We have just verified that $\delta_Y \circ a$ is non-zero and, hence, injective. A diagram chase then implies that δ_X is injective and, hence, bijective. Another diagram chase then implies that b is injective. By Proposition 5.6.1, we are done. □

Remark 5.6.3. Consider the special case of Proposition 5.6.2 where all the line bundles occuring in the $\mathcal{E}_i$ are antiample. Since X is Gorenstein, one may be tempted to say that the given proof of Proposition 5.6.2 goes through without using Proposition 5.5.3 as we can simply use Frobenius to kill cohomology after dualising. However, this is false: we applied Proposition 5.5.3 to finite covers $Y \to X$ rather than X itself, and there is no reason we can suppose that Y is Gorenstein. If we alter Y to a Gorenstein (or even regular) scheme, then we lose ampleness, and are once again in a position where we need to use Proposition 5.5.3.

Remark 5.6.4. The assumptions in Proposition 5.6.2 are extremely strong. Consider the special case where $Z \hookrightarrow X$ is a divisor. In this case, the natural resolution (and, in fact, the only available one) to consider is:

$$[\omega_X \to \omega_X(Z)] \simeq \omega_Z$$

The assumptions of Proposition 5.6.2 will be satisfied precisely when $\omega_X^{-1}(-Z)$ is semiample and big. This implies that ω_X^{-1} is also big. In particular, X is birationally Fano.

Proposition 5.6.2 looks slightly bizarre at first glance. However, it is a useful argument in inductive proofs whenever one wants to translate the property of having Condition 1.0.2 from a subvariety to the total space. Here is a typical application:

Proposition 5.6.5. *Let V be a vector space of dimension d over a field k, and let $\mathrm{Flag}(V)$ be the moduli space of complete flags $(0 = F_0 \subset F_1 \subset \cdots F_{d-1} \subset F_d = V)$ in V. Then $\mathrm{Flag}(V)$ satisfies Condition 1.0.2.*

Proof. We work by induction on the dimension d. The case $d = 0$ being trivial, we may assume that $\mathrm{Flag}(W)$ satisfies Condition 1.0.2 for any vector space W of dimension $\leq d - 1$. If we let $\mathbf{P}(V)$ denote the projective space of *hyperplanes* in V, then there is a natural morphism $\pi : \mathrm{Flag}(V) \to \mathbf{P}(V)$ given by sending a complete flag $(0 = F_0 \subset F_1 \subset \cdots F_{d-1} \subset F_d = V)$ to the hyperplane $(F_{d-1} \subset V)$. The morphism π can easily be checked to be projective and smooth. Let $W \subset V$ be a fixed hyperplane, and let $b \in \mathbf{P}(V)(k)$ be the corresponding point. The fibre $\pi^{-1}(b)$ is identified with $\mathrm{Flag}(W)$. We will apply Proposition 5.6.2 with $Z = \mathrm{Flag}(W)$ and $X = \mathrm{Flag}(V)$ to get the desired result.

The structure sheaf $\kappa(b)$ of the point $b : \mathrm{Spec}(k) \hookrightarrow \mathbf{P}(V)$ can be realised as the zero locus of a section of $\mathcal{O}(1)^{\oplus(d-1)}$ by thinking of b as the intersection of $(d-1)$ hyperplanes in general position. This gives us a Koszul resolution

$$[\mathcal{O}(-(d-1)) \simeq \wedge^{d-1}(\mathcal{O}(-1)^{\oplus(d-1)}) \to \cdots \to \mathcal{O}(-1)^{\oplus(d-1)} \to \mathcal{O}] \simeq \kappa(b).$$

Twisting by $\mathcal{O}(-1)$, we find a resolution

$$[\omega_{\mathbf{P}(V)} \to \mathcal{M}_{d-2} \to \cdots \to \mathcal{M}_1 \to \mathcal{M}_0] \simeq \kappa(b)$$

with each $\mathcal{M}_i$ a direct sum of inverses of ample line bundles with degrees between 1 and $d-2$. Pulling this data back along π, we find a resolution

$$[\pi^*\omega_{\mathbf{P}(V)} \to \pi^*\mathcal{M}_{d-2} \to \cdots \to \pi^*\mathcal{M}_1 \to \pi^*\mathcal{M}_0] \simeq \pi^*\kappa(b) = \mathcal{O}_Z.$$

Twisting by the relative dualising sheaf ω_π, we find

$$[\omega_\pi \otimes \pi^*\omega_{\mathbf{P}(V)} \to \omega_\pi \otimes \pi^*\mathcal{M}_{d-2} \to \cdots \to \omega_\pi \otimes \pi^*\mathcal{M}_1 \to \omega_\pi \otimes \pi^*\mathcal{M}_0] \simeq \omega_\pi|_Z.$$

Since π is smooth, we identify $\omega_X \simeq \omega_\pi \otimes \pi^*\omega_{\mathbf{P}(V)}$, and $\omega_Z \simeq \omega_\pi|_Z$. Thus, we obtain a resolution

$$[\omega_X \to \omega_\pi \otimes \pi^*\mathcal{M}_{d-2} \to \cdots \to \omega_\pi \otimes \pi^*\mathcal{M}_1 \to \omega_\pi \otimes \pi^*\mathcal{M}_0] \simeq \omega_Z$$

with $\mathcal{M}_i$ as above. Standard calculations with flag varieties (see Lemma 5.6.6) now show that the terms $\omega_\pi \otimes \pi^*\mathcal{M}_i$ are direct sums of inverses of semiample and big line bundles. In particular, this resolution has the form required in Propositon 5.6.2. Hence, we win by induction. ☐

We needed to calculate the positivity of certain natural line bundles on the flag variety in Proposition 5.6.5. Since we were unable to find a satisfactory reference, we carry out the calculation here.

Lemma 5.6.6. *Let V be an n-dimensional vector space over a field k, let $\pi : \mathrm{Flag}(V) \to \mathbf{P}(V)$ be the natural morphism. For all $i > 0$ and all n, the line bundles $\omega_\pi \otimes \pi^*\mathcal{O}(-i)$ are inverses of semiample and big line bundles.*

Proof. For $n = 2$, the map π is an isomorphism, and the claim is obvious. Assume $n \geq 3$. Let

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = V \otimes \mathcal{O}_{\mathrm{Flag}(V)}$$

be the universal flag on $\mathrm{Flag}(V)$ with $\dim(V_i) = i$. For each $i \geq 1$, let $\mathcal{L}_i = \mathcal{V}_i/\mathcal{V}_{i-1}$ be the associated line bundle. The tangent bundle of $\mathrm{Flag}(V)$ admits a filtration whose pieces are of the form

$$\mathcal{H}\mathrm{om}(\mathcal{V}_i, \mathcal{L}_{i+1}) \simeq \mathcal{V}_i^\vee \otimes \mathcal{L}_{i+1}$$

for $1 \leq i \leq n-1$. This filtration gives us the formula

$$\omega^{-1}_{\mathrm{Flag}(V)} \simeq \otimes_{i=1}^{n-1}(\det(\mathcal{V}_i)^{-1} \otimes \det(\mathcal{L}_{i+1})^i).$$

Since each $\mathcal{V}_i$ is filtered with pieces of the form $\mathcal{L}_j$ for $1 \leq j \leq i$, we find

$$\omega^{-1}_{\mathrm{Flag}(V)} \simeq \otimes_{i=1}^{n-1}(\mathcal{L}_1^{-1} \otimes \mathcal{L}_2^{-1} \otimes \cdots \otimes \mathcal{L}_i^{-1} \otimes \mathcal{L}_{i+1}^i).$$

Collecting terms, we find

$$\omega^{-1}_{\mathrm{Flag}(V)} \simeq (\otimes_{i=1}^{n-1}\mathcal{L}_i^{2i-n}) \otimes \mathcal{L}_n^{n-1}.$$

The inverse $\mathcal{M}_j$ of the line bundle $\omega_\pi \otimes \pi^*\mathcal{O}(-j)$ can be written as

$$\mathcal{M}_j \simeq \omega_\pi^{-1} \otimes \pi^*\mathcal{O}(j) \simeq \omega^{-1}_{\mathrm{Flag}(V)} \otimes \pi^*(\omega_{\mathbf{P}(V)} \otimes \mathcal{O}(j)).$$

Using the formula for $\omega^{-1}_{\mathrm{Flag}(V)}$ we arrived at earlier, and the fact that π is defined by the tautological quotient $\mathcal{O}_{\mathrm{Flag}(V)} \otimes V \twoheadrightarrow \mathcal{L}_n$, we can simplify the preceding formula to get

$$\mathcal{M}_j \simeq (\otimes_{i=1}^{n-1}\mathcal{L}_i^{2i-n}) \otimes \mathcal{L}_n^{n-1} \otimes \mathcal{L}_n^{-n+j} \simeq (\otimes_{i=1}^{n-1}\mathcal{L}_i^{2i-n}) \otimes \mathcal{L}_n^{j-1}.$$

Our goal is to show that $\mathcal{M}_j$ is semiample and big for $j > 0$. Being the pullback of a very ample line bundle, the factor $\mathcal{L}_n^{j-1}$ is semiample and effective for $j > 0$. Hence, it suffices to show that

$$\mathcal{N} := \otimes_{i=1}^{2i-n}\mathcal{L}_i^{2i-n}$$

is semiample and big. Since we have assumed that $n \geq 3$, the center $c = \lfloor \frac{n-1}{2} \rfloor$ is strictly positive. We may then write

$$\mathcal{N} \simeq \otimes_{k=1}^{c}(\mathcal{L}_{n-k} \otimes \mathcal{L}_k^{-1})^{\otimes(n-2k)}.$$

Schubert calculus (see [Ful97, §10.2, Proposition 3]) tells us that the line bundles $\mathcal{L}_a \otimes \mathcal{L}_b^{-1}$ are ample when $a > b$. In particular, all the factors in the preceding factorisation of $\mathcal{N}$ are ample. Since $c \geq 1$, this factorisation is also non-empty. It follows then that $\mathcal{N}$ is an ample line bundle, as desired. □

Remark 5.6.7. Proposition 5.6.5 can be improved slightly to say that $\omega_\pi \otimes \pi^*\mathcal{O}(-i)$ is actually the inverse of an ample line bundle. This claim follows directly from the homogeneity of $\mathrm{Flag}(V)$. Indeed, let $\mathcal{L}$ be a semiample and big line bundle on a projective variety X that is homogeneous for a connected group G. Let $f : X \to \mathbf{P}^N$ denote the map defined by a suitably large power of $\mathcal{L}$. If $\mathcal{L}$ was not ample, then there would be a proper curve $C \subset X$ that is contracted by f. By the rigidity lemma (see [MFK94, Proposition 6.1]), the same is true for any curve algebraically equivalent to C. However, since X is homogeneous, translates of C under G actually cover X. Since G is connected, all translates of C are algebraically equivalent to C. It follows then that $\dim(\mathrm{im}(f)) < \dim(X)$ contradicting the bigness of $\mathcal{L}$.

Remark 5.6.8. Proposition 5.6.5 asserts that G/B satisfies Condition 1.0.2 when $G = \mathrm{GL}(V)$, and $B \subset G$ is the standard Borel subgroup. Similar arguments to the ones given above should also work when G is the orthogonal group associated to a quadratic form (V, q), though we have not checked that. In fact, it seems entirely plausible that the above arguments can be made to show that Condition 1.0.2 is verified by G/B for any algebraic group G. The idea would be to show that Condition 1.0.2 is satisfied by Bott-Samelson variety X for G (see [BS55]). As X admits a proper birational map $\pi : X \to G/B$ satisfying $\mathcal{O}_{G/B} \simeq \mathrm{R}\pi_*\mathcal{O}_X$, the validity of Condition 1.0.2 for X implies that for G/B. To show it for X, one would use that X comes equipped with a natural structure an explicit iterated $\mathbf{P}^1$-bundle with sections $X = X_n \to X_{n-1} \to \cdots \to X_1 \simeq \mathbf{P}^1 \to X_0 \simeq *$.

As a corollary, we obtain a further family of examples.

Corollary 5.6.9. *Let V be a finite dimensional vector space, and let X be a partial Flag variety for V. Then X satisfies Condition 1.0.2. In particular, all Grassmanians $\mathrm{Gr}(k, n)$ satisfy Condition 1.0.2.*

Proof. There is a natural morphism $\pi : \mathrm{Flag}(V) \to X$ given by remembering the corresponding flag. It can be checked that π is a smooth projective morphism whose fibres are iterated fibrations of projective spaces. In particular, $\mathrm{R}\pi_*\mathcal{O}_{\mathrm{Flag}(V)} \simeq \pi_*\mathcal{O}_{\mathrm{Flag}(V)} \simeq \mathcal{O}_X$. The result now follows from Lemma 3.1.3. □

Remark 5.6.10. Proposition 5.6.5 and Corollary 5.6.9 imply, in particular, that a (partial) flag variety is Frobenius split. The proof presented above seems to be qualitatively different proof than the standard proofs.

Chapter 6

Some results on group schemes

All group schemes occuring in this chapter are commutative; all the cohomology groups occuring in this chapter are computed in the fppf topology unless otherwise specified. Our primary goal is to prove the following theorems on the cohomology of group schemes:

Theorem 6.0.1. *Let S be a noetherian excellent scheme, and let G be a finite flat commutative group scheme over S. Then classes in $H^n(S, G)$ can be killed by finite surjective maps for $n > 0$.*

Theorem 6.0.2. *Let S be a noetherian excellent scheme, and let A be an abelian scheme over S. Then classes in $H^n(S, A)$ can be killed by proper surjective maps for $n > 0$.*

We stress that there are no assumptions on the residue characteristics of the base scheme S in Theorems 6.0.1 and 6.0.2. The plan for this chapter is as follows. In §6.1 we recall an observation originally due to Gabber concerning the local structure of the étale topology. Using this observation, we prove Theorem 6.0.1 in §6.2, and Theorem 6.0.2 in §6.3. Next, in §6.4, we explain how to use Theorem 6.0.1 to give a new and more conceptual proof of Theorem 5.0.1, the main theorem of Chapter 5; this was the primary motivation for most results in this chapter. We close in §6.5 by giving an example illustrating the necessity of "proper" in Theorem 6.0.2.

6.1 An observation of Gabber

In this section we recall an observation due to Gabber concerning the étale topology. The utility of this observation to us is that it permits reduction of étale cohomological considerations to those in finite flat cohomology and those in Zariski cohomology.

Lemma 6.1.1 (Gabber, [Hoo82, Lemma 5])**.** *Let $f : U \to X$ be a surjective étale morphism of affine schemes. Then there exists a finite flat map $g : X' \to X$, and a Zariski open cover $\{U_i \hookrightarrow X'\}$ such that the natural map $\sqcup_i U_i \to X$ factors through $U \to X$.*

For the convenience of the reader, we sketch a proof.

Sketch of proof. We first explain how to deal with the local case. Assume that $X = \mathrm{Spec}(A)$ is the spectrum of a local ring A, and $U = \mathrm{Spec}(B)$ is the spectrum of a local étale A-algebra B. The structure theorem for étale morphisms (see [Gro03, Exposé I, Théorème 7.6]) implies that $B = C_{\mathfrak{m}}$ where $C = A[x]/(f(x))$ with $f(x) = x^n + a_1x^{n-1} + \cdots + a_n$ a monic polynomial, and $\mathfrak{m} \subset C$ a maximal ideal with $f'(x) \notin \mathfrak{m}$. We define

$$D = A[x_1, \cdots, x_n]/(\sigma_i(x_1, \cdots, x_n) - (-1)^{n-i}a_i)$$

where $\sigma_1, \cdots, \sigma_n$ are the elementary symmetric polynomials in the x_i's. This ring is finite free over A of rank $n!$, admits an action of S_n that is transitive on the maximal ideals, and formalises the idea that the coefficients of $f(x)$ can be written as elementary symmetric functions in its roots. In particular, there is a natural morphism $C \to D$ sending x to x_1. As both C and D are finite free over A, there is a maximal ideal $\mathfrak{m}_1 \subset D$ lying over $\mathfrak{m} \subset C$. Thus, there is a natural map $a : B \to D_{\mathfrak{m}_1}$. By the S_n-action, for every maximal ideal $\mathfrak{n} \subset D$, there is an automorphism $D \to D$ sending $\mathfrak{m}_1$ to $\mathfrak{n}$. Composing such an automorphism with a, we see that for every maximal ideal $\mathfrak{n} \subset D$, the structure map $A \to D_\mathfrak{n}$ factorises through $A \to B$ for some map $B \to D_\mathfrak{n}$; the claim follows. In general, one reduces to the local case by considering fibre products as in Proposition 5.2.1 □

Actually, we use a slight weakening of Gabber's result – relaxing finite flat to finite surjective – that remains true when the schemes under consideration are no longer assumed to be affine.

Lemma 6.1.2. *Let $f : U \to X$ be a surjective étale morphism of schemes. Then there exists a finite surjective map $g : X' \to X$, and a Zariski open cover $\{U_i \hookrightarrow X'\}$ such that the natural map $\sqcup_i U_i \to X$ factors through $U \to X$.*

Proof. We can solve the problem locally on X by Lemma 6.1.1. This means that there exists a Zariski open cover $\{V_i \hookrightarrow X\}$, finite surjective maps $W_i \to V_i$, and Zariski covers $\{Y_{ij} \hookrightarrow W_i\}$ such that $\sqcup Y_{ij} \to V_i$ factors throughts $U \times_X V_i \to V_i$. By Proposition 5.2.1, we may find a single finite surjective map $W \to X$ such that $W \times_X V_i \to V_i$ factors through $W_i \to V_i$. Setting $X' = W$ and pulling back the covers $\{Y_{ij} \to W_i\}$ to $W \times_X V_i$ then solves the problem. □

6.2 The theorem for finite flat commutative group schemes

In this section we prove Theorem 6.0.1. Roughly speaking, our strategy is to first use theorems of Raynaud and Grothendieck to reduce to étale cohomology from flat cohomology, then use Gabber's observation from §6.1 to reduce to Zariski cohomology, and then solve the problem by hand. To carry this program out, we now explain how to reduce the fppf cohomology of finite flat group schemes to étale cohomology; it turns out that they are almost the same.

Proposition 6.2.1. *Let S be the spectrum of a strictly henselian local ring, and let G be a finite flat commutative group scheme over S. Then $H^i(S, G) = 0$ for $i > 1$.*

Proof. We first explain the idea informally. Using a theorem of Raynaud, we can embed G into an abelian scheme, which allows us to express the cohomology of G in terms of that of abelian schemes. As abelian schemes are smooth, a result of Grothendieck ensures that their fppf cohomology coincides with their étale cohomology. As the latter vanishes when S is strictly henselian, we obtain the desired conclusion.

Now for the details: by a theorem of Raynaud (see [BBM82, Théorème 3.1.1]), there exists an abelian scheme $A \to S$ and an S-closed immersion $G \hookrightarrow A$ of group schemes. Let A/G denote the quotient stack in the fppf topology. It is well known that A/G is an abelian scheme over S, but we were unable to find a reference, so we give a proof. By a theorem of Michael Artin (see [Art74, Corollary 6.3]), the quotient A/G is an algebraic stack. Since the action of G on A is free, the quotient is actually a discrete algebraic stack. As explained in [LMB00, Corollary 10.4], it follows that A/G is actually an algebraic space. The fact that A and G are locally of finite presentation implies the same is true for A/G. Moreover, the quotient $A \to A/G$ is flat by construction. The Auslander-Buschbaum theorem then forces the geometric fibres of $A/G \to S$ to be regular. By the fibre-by-fibre smoothness criterion (see [Gro67, Théorème 17.5.1]), it follows that $A/G \to S$ is smooth. Since $A \to A/G$ is surjective, we also know that $A/G \to S$ is proper with geometrically connected fibres. The group structure on A descends to give A/G the structure of an abelian sheaf relative to S. Hence, we find that A/G is an algebraic space over S representing a sheaf of abelian groups with the additional property that the structure morphism $A/G \to S$ is a proper smooth morphism with geometrically connected fibres, i.e., the space $A/G \to S$ is an abelian algebraic space. By a different theorem of Raynaud (see [FC90, Theorem 1.9]), it follows that A/G is actually an abelian scheme over S. Hence, we have a short exact sequence

$$0 \to G \to A \to A/G \to 0$$

of sheaves on the fppf site of S relating the finite flat commutative group scheme G to the abelian schemes A and A/G. This gives rise to a long exact sequence

$$\cdots H^{n-1}(S, A/G) \to H^n(S, G) \to H^n(S, A) \to \cdots$$

of cohomology groups. By Grothendieck's theorem (see [Gro68b, Théorème 11.7]), fppf cohomology coincides with étale cohomology when the coefficients are smooth group schemes. In particular, this applies to A and A/G. As S is strictly henselian, it follows that $H^i(S, A) = H^i(S, A/G) = 0$ for $i > 0$. The claim about G now follows from the preceding exact sequence. □

Next, we explain how to deal with Zariski cohomology with coefficients in a finite flat commutative group scheme.

Proposition 6.2.2. *Let S be a normal noetherian scheme, and let $G \to S$ be a finite flat commutative group scheme. Then $H^n_{\mathrm{Zar}}(S, G) = 0$ for $n > 0$.*

Proof. We may assume that S is connected. As constant sheaves on irreducible topological spaces are acyclic, it will suffice to show that G restricts to a constant sheaf on the small Zariski site of S, i.e., that the restriction maps $G(S) \to G(U)$ are bijective for any non-empty open subset $U \hookrightarrow S$. Injectivity follows from the density of $U \hookrightarrow S$ and the separatedness of $G \to S$. To show surjectivity, we note that given a section $U \to G$ of G over U, we can simply take the scheme-theoretic closure of U in G to obtain an integral closed subscheme $S' \hookrightarrow G$ such that the projection map $S' \to S$ is finite and an isomorphism over U. By the normality of S, this forces $S' = S$. Thus, G restricts to the constant sheaf on $G(S)$ as claimed. □

We can now complete the proof of Theorem 6.0.1 by following the outline sketched earlier.

Proof of Theorem 6.0.1. Let S be a noetherian excellent $\mathbf{F}_p$-scheme, let $G \to S$ be a finite flat commutative group scheme. We need to show that classes in $H^n(S, G)$ can be killed by finite covers for S for $n > 0$. We deal with the $n = 1$ case on its own, and then proceed inductively.

For $n = 1$, note that classes in $H^1(S, G)$ are represented by fppf G-torsors T over S. By faithfully flat descent for finite flat morphisms, such schemes $T \to S$ are also finite flat. Passing to the total space of T trivialises the G-torsor T. Therefore, classes in $H^1(S, G)$ can be killed by finite flat covers of S.

We now fix an integer $n > 1$ and a cohomology class $\alpha \in H^n(S, G)$. By Proposition 6.2.1, we know that there exists an étale cover of S over which α trivialises. By Lemma 6.1.2, after replacing S by a finite cover, may assume that there exists a Zariski cover $\mathfrak{U} = \{U_i \hookrightarrow S\}$ such that $\alpha|_{U_i}$ is Zariski locally trivial. The Cech spectral sequence for this cover is

$$H^p(\mathfrak{U}, H^q(G)) \Rightarrow H^{p+q}(S, G)$$

where $H^q(G)$ is the Zariski presheaf $V \mapsto H^q(V, G)$. By construction, the class α comes from some $\alpha' \in H^{n-q}(\mathfrak{U}, H^q(G))$ with $q < n$. The group $H^{n-q}(\mathfrak{U}, H^q(G))$ is the $(n-q)$-th cohomology group of the standard Cech complex

$$\prod_i H^q(U_i, G) \to \prod_{i<j} H^q(U_{ij}, G) \to \dots$$

By the inductive assumption and the fact that $q < n$, terms of this complex can be annihilated by finite covers of the corresponding schemes. By Proposition 5.2.1, we may refine these finite covers by one that comes from all of S. In other words, we can find a finite surjective cover $S' \to S$ such that $\alpha'|_{S'} = 0$. After replacing S with S', the Cech spectral sequence then implies that α comes from some $H^{n-q'}(\mathfrak{U}, H^{q'}(G))$ with $q' < q$. Proceeding in this manner, we can reduce the second index q all the way down to 0, i.e., assume that the class α lies in the image of the map

$$H^n(\mathfrak{U}, G) \to H^n(S, G).$$

Now we are reduced to the situation in Zariski cohomology that was tackled in Proposition 6.2.2. □

Remark 6.2.3. The proof given above for Theorem 6.0.1 used the intermediary of abelian schemes to make the connection between fppf cohomology and étale cohomology with coefficients in a finite flat group commutative scheme G (see Proposition 6.2.1). When the coefficient group scheme G is smooth (or equivalently étale), this reduction can easily be avoided. In particular, if one follows the arguments we give in §6.4, then it is possible give a relatively elementary proof of Theorem 5.0.1 using the simple version with étale coefficients provided the base S is a field.

6.3 The theorem for abelian schemes

Our goal in this section is to prove Theorem 6.0.2. The arguments here essentially mirror those for finite flat commutative group schemes presented in §6.2. The key difference is that annihilating Zariski cohomology requires more complicated constructions when the coefficients are abelian schemes. We handle this by proving a generalisation of Weil's extension lemma (see Proposition 6.3.3). This generalisation requires strong regularity assumptions on S and is one of the two places in our proof of Theorem 6.0.2 that we need *proper* covers instead of finite ones; the other is the case of H^1.

We begin by recording an elementary criterion for a map to an abelian variety to be constant.

Lemma 6.3.1. *Let A be an abelian variety over an algebraically closed field k, and let C be a reduced variety over k. Fix an integer ℓ invertible on k. A map $g : C \to A$ is constant if and only if it induces the* 0 *map $H^1_{\text{ét}}(A, \mathbf{Q}_\ell) \to H^1_{\text{ét}}(C, \mathbf{Q}_\ell)$.*

Proof. It suffices to show that a map like g that induces the 0 map on H^1 is trivial. As any k-variety is covered by curves, it suffices to show that the map g is constant on all curves in C. Thus, we reduce to the case that C is a curve. We may also clearly assume that C is normal, i.e., smooth. Let $\overline{C}$ denote the canonical smooth projective model of C. Since A is proper, the map g factors through a map $\overline{g} : \overline{C} \to A$. Since C and $\overline{C}$ are normal, the map $\pi_1(C) \to \pi_1(\overline{C})$ is surjective. Hence, the map $H^1_{\text{ét}}(\overline{C}, \mathbf{Q}_\ell) \to H^1_{\text{ét}}(C, \mathbf{Q}_\ell)$ is injective. Thus, to answer the question, we may assume that $C = \overline{C}$ is a smooth projective curve.

Let $A \hookrightarrow \mathbf{P}^n$ be a closed immersion corresponding to a very ample line bundle $\mathcal{L}$. The map $g : C \to A$ will be constant if we can show that $g^*\mathcal{L}$ is not ample, i.e., has degree 0.

As the ℓ-adic cohomology of an abelian variety is generated[1] in degree 1, the hypothesis on H^1 implies that the map $H^2_{\text{ét}}(A, \mathbf{Q}_\ell) \to H^2(C, \mathbf{Q}_\ell)$ is also 0. In particular, $g^*(c_1\mathcal{L}) = 0$, where $c_1(\mathcal{L}) \in H^2(A, \mathbf{Q}_\ell(1)) \simeq H^2(A, \mathbf{Q}_\ell)$ is the first Chern class of the line bundle $\mathcal{L}$. Since applying g^* commutes with taking the first Chern class, it follows that $c_1(g^*\mathcal{L}) = 0$, hence $g^*\mathcal{L}$ has degree 0 as desired. □

Remark 6.3.2. If we were working over $\mathbf{C}$, then part of Lemma 6.3.1 can be proven easily using arguments from topology. As an abelian variety A is a $K(\pi_1(A), 1)$, homotopy classes of maps $X \to A$ are in bijective correspondence with cohomology classes $H^1(X, \pi_1(A))$ for any CW complex X. In particular, such a map is homotopic to a constant map if the induced map on H^1's is 0. One then needs to show that a map between projective varieties inducing the 0 map on (Betti) cohomology is actually constant; this is essentially what is proven above.

We now prove the promised extension theorem for maps into abelian schemes.

Proposition 6.3.3. *Let S be a regular connected excellent noetherian scheme, and let $f : A \to S$ be an abelian scheme. For any non-empty open $j : U \hookrightarrow S$, the natural restriction map $A(S) \to A(U)$ is bijective*[2].

Proof. The bijectivity of $A(S) \to A(U)$ will follow by taking global sections if we can show that the natural map of presheaves $a : A \to j_*(A|_U)$ is an isomorphism on the small Zariski site of S. As both the source and the target of a are actually sheaves for the étale topology on S, we may localise to assume that S is the spectrum of a strictly henselian local ring R. In this setting, we will show that $A(S) \to A(U)$ is bijective using ℓ-adic cohomology.

The injectivity of $A(S) \to A(U)$ follows from the fact that $U \hookrightarrow S$ is dense, and that $A \to S$ is separated. To show surjectivity, by the valuative criterion of properness, we may assume that U contains all the codimension 1 points, i.e, the complement $S \setminus U$ has codimension at least 2 in S. Let $s : U \to A$ be a section of A over U. By taking the normalised scheme-theoretic closure of U in A, we obtain a proper birational map $p : S' \to S$ that is an isomorphism over U, and an S-map $i : S' \to A$ extending the given section over U. The desired surjectivity claim will follow if we can show that i is constant on the fibres of p. Since $p_*\mathcal{O}_{S'} = \mathcal{O}_S$, it even suffices to show that i collapses the reduced schemes underlying the fibres of p. As p is proper, by semicontinuity of dimension, it suffices to show that i collapses the reduced special fibre S'_s, where $s \in S$ is the closed point. By Lemma 6.3.1, it suffices to show that the induced map $H^1(A_s, \mathbf{Q}_\ell) \to H^1(S'_s, \mathbf{Q}_\ell)$ is trivial for

[1] We were unable to find a reference that proves the generation in degree 1 directly. For completeness, we sketch a proof. If the base field is the field $\mathbf{C}$ of complex numbers, then Artin's comparison theorem [SGA73, Théorème 4.4, Exposé XI] reduces the calculation to the singular cohomology of the complex manifold $A(\mathbf{C})$. Since A can be uniformised, we see that $A(\mathbf{C}) \simeq (S^1)^{2g}$ as topological spaces, where $g = \dim(X)$. The Kunneth formula implies that $\mathrm{R}\Gamma(A(\mathbf{C}), \mathbf{Z}) \simeq \mathrm{R}\Gamma(S^1, \mathbf{Z})^{\otimes 2g}$. As the cohomology of the circle is torsion free and concentrated in degrees 0 and 1, the claim follows. If the base field k has characteristic 0, we may replace it with a smaller algebraically closed field that embeds into $\mathbf{C}$ using [Del77, Arcata, V, Corollaire 3.3], and then pass to $\mathbf{C}$ using [Del77, Arcata, V, Corollaire 3.3] again, whence the desired claim follows. If k has positive characteristic p, then the main theorem of [Mum69] implies that A lifts to a smooth projective morphism $\mathcal{A} \to \mathrm{Spec}(R)$, where R is a p-adic discrete valuation with residue field k and fraction field K of characteristic 0. The proper and smooth base change theorems imply (see [Del77, Arcata, VI, §4]) that $H^i(A_{\overline{K}}, \mathbf{Q}_\ell) \simeq H^i(A_k, \mathbf{Q}_\ell)$, and the claim now follows from the characteristic 0 version.

[2] Professor János Kollár pointed out to the author, after the present work was completed, that this claim also follows from a theorem of Abhyankar as presented in [Kol96, §VI.1, Theorem 1.2]. Abhyankar's theorem implies that for any proper modification $p : S' \to S$ with S noetherian regular excellent, the positive dimensional fibres of p contain non-constant rational curves. Applying this theorem to the graph of a rational map defined by a section $U \to A$ over an open $U \subset S$ gives our desired claim as abelian varieties do not contain rational curves. We would like to thank Professor Kollár for pointing this out. As the proof given above is different and simpler, we include it anyways.

some integer ℓ invertible on S. Note that we have the following commutative diagram:

$$\begin{array}{ccc} H^1(A, \mathbf{Q}_\ell) & \longrightarrow & H^1(A_s, \mathbf{Q}_\ell) \\ \downarrow & & \downarrow \\ H^1(S', \mathbf{Q}_\ell) & \longrightarrow & H^1(S'_s, \mathbf{Q}_\ell). \end{array}$$

The horizontal maps are isomorphisms by the proper base change theorem in étale cohomology (see [Del77, Arcata IV-1, Théorème 1.2]) as S is a strictly henselian local scheme. Hence, it suffices to show that $H^1(A, \mathbf{Q}_\ell) \to H^1(S', \mathbf{Q}_\ell)$ is 0. This will follow if we can show that $\pi_1(S') \to \pi_1(A)$ is the 0 map. As S' is normal, we know that $\pi_1(U) \twoheadrightarrow \pi_1(S')$ is surjective. Thus, it suffices to show that $\pi_1(U) = 0$. By Zariski-Nagata purity (see [Gro68a, Exposé X, Théorème 3.4]) and the fact that U has codimension at least 2 in S, we know that $\pi_1(U) \simeq \pi_1(S)$. Since S is the spectrum of a strictly henselian regular local ring, we have that $\pi_1(S) = 0$ and, therefore, $\pi_1(U) = 0$ as desired. □

Remark 6.3.4. The main idea for the proof of Proposition 6.3.3 comes from topology. The stack $\mathcal{A}_g$ of abelian varieties is an Eilenberg-Maclane space. Since an abelian variety is also an Eilenberg-Maclane space, the total space $\mathcal{U}_g$ of the universal family $\mathcal{U}_g \to \mathcal{A}_g$ of abelian varieties is also an Eilenberg-Maclane space. Proposition 6.3.3 can then be rephrased as asking if every map $U \to \mathcal{U}_g$ with a specified extension $S \to \mathcal{A}_g$ extends to a map $S \to \mathcal{U}_g$ provided S is smooth, and $U \subset S$ is a dense open subset. At the level of homotopy types, the answer would be yes if we could show that the map $\pi_1(U) \to \pi_1(\mathcal{U}_g)$ factors through a map $\pi_1(S) \to \pi_1(\mathcal{U}_g)$. This is essentially what is verified above using purity; Lemma 6.3.1 allows us to go from this homotopy-theoretic conclusion to a geometric one.

Remark 6.3.5. Proposition 6.3.3 can be considered a generalisation of Weil's extension lemma when applied to abelian varieties. Recall that this lemma says that the domain of definition of rational maps from a smooth variety to a group variety has pure codimension 1. In case the target is proper, i.e., an abelian variety A, this reduces to the statement that $A(X) \simeq A(U)$ for any smooth variety X, and dense open $U \hookrightarrow X$.

Remark 6.3.6. Our proof of Proposition 6.3.3 is an ℓ-adic (or equivalently, topological) one. One can given a more geometric and, perhaps more conceptual, proof in characteristic 0 as follows. Assume that S is a smooth affine variety, and that $U \hookrightarrow S$ is a dense open subset with codimension at least 2. Given a section $s : U \to A$ of A over U, one can modify s and use resolution of singularities to obtain a smooth variety S', a birational morphism $p : S' \to S$ that is an isomorphism over U, and a map $i : S' \to A$ extending s over U. Since S is smooth, it has rational singularities. In particular, we have $\mathcal{O}_S \simeq \mathrm{R}p_*\mathcal{O}_{S'}$. It then follows that for any fibre F of p, the induced map $H^i(A, \mathcal{O}_A) \to H^i(F, \mathcal{O}_F)$ is 0. An analogue of the argument presented in Lemma 6.3.1 then finishes the job (this uses characteristic 0 once again). An advantage of this argument is that it works as long as S has rational singularities. It also suggests a question to which we do not know the answer: if S is a scheme in positive characteristic satisfying some definition of rational singularities (such as Condition 1.0.2, or F-rationality, or pseudorationality), does Proposition 6.3.3 hold for S?

Example 6.3.7. We give an example to show that the regularity condition on S cannot be weakened too much in Proposition 6.3.3. Let $(E, e) \subset \mathbf{P}^2$ be an elliptic curve, and let S be the affine cone on E with origin s. Note that S is a hypersurface singularity of dimension 2 with 0 dimensional singular locus. In particular, it is normal. Let $A = S \times E$ denote the constant abelian scheme on E

over S. Then $U = S \setminus \{s\}$ can be identified with the total space of the $\mathbf{G}_m$-torsor $\mathcal{O}(-1)|_E - 0(E)$ over E. Thus, there exists a non-constant section of $A(U)$. On the other hand, all sections $S \to A$ are constant. Indeed, every point in S lies on an $\mathbf{A}^1$ containing s. As all maps $\mathbf{A}^1 \to E$ are constant, the claim follows. Thus, we obtain an example of a normal hypersurface singularity S and an abelian scheme $A \to S$ such that the conclusion of Proposition 6.3.3 fails for S. Of course, S is not a rational singularity, a fact supported by Remark 6.3.6.

Next, we point out how to use Proposition 6.3.3 to prove the version of Theorem 6.0.2 involving Zariski cohomology under strong regularity assumptions on the base scheme S; the proof is trivial.

Corollary 6.3.8. *Let S be a regular excellent noetherian scheme, and let $f : A \to S$ be an abelian scheme. Then $H^n_{\mathrm{Zar}}(S, A) = 0$ for $n > 0$.*

Proof. By Proposition 6.3.3, we know that A restricts to a constant sheaf on the small Zariski site of each connected component of S. By the vanishing of the cohomology of a constant sheaf on an irreducible topological space, the claim follows. ☐

We are now in a position to complete the proof of Theorem 6.0.2.

Proof of Theorem 6.0.2. Let S be a noetherian excellent scheme, and let $A \to S$ be an abelian scheme. We will show that that cohomology classes in $H^n(S, A)$ are killed by proper surjective maps by induction on n provided $n > 0$. We may assume that S is integral.

For $n = 1$, classes in $H^1(S, A)$ are represented by étale A-torsors T over S. As T is an fppf S-scheme, there exists a quasi-finite dominant morphism $U \to S$ such that $T(U)$ is non-empty. By picking an S-map $U \to T$ and taking the closure of the image, we obtain a proper surjective cover $S' \to S$ such that $T(S')$ is not empty. This implies that the cohomology class associated to T dies on passage to S', proving the claim.

We next proceed exactly as in the proof of Theorem 6.0.1 to reduce down to the case of Zariski cohomology. The only difference is that the references to Proposition 6.2.1 are replaced by references to Grothendieck's theorem (see [Gro68b, Théorème 11.7]) which, in particular, implies that cohomology classes in $H^n(S, A)$ trivialise over an étale cover; we omit the details.

To show the claim for Zariski cohomology, assume first that S is of finite type over $\mathbf{Z}$. In this case, thanks to de Jong's theorems from [dJ97], we can find a proper surjective cover of S with regular total space. Passing to this cover and applying Corollary 6.3.8 then solves the problem. In the case that S is no longer of finite type over $\mathbf{Z}$, we reduce to the finite type case using approximation. Indeed, the data (S, A, α) comprising of the base scheme S, the abelian scheme $A \to S$, and a Zariski cohomology class $\alpha \in H^n_{\mathrm{Zar}}(S, A)$ can be approximated by similar data with all schemes involved of finite type over $\mathbf{Z}$. Given such an approximating triple (S', A', α') with S' of finite type over $\mathbf{Z}$, we can find a proper surjective map $S'' \to S'$ killing α' by the earlier argument. By functoriality, the pullback $S'' \times_{S'} S \to S$ is a proper surjective cover of S killing α. ☐

6.4 The alternative proof of Theorem 5.0.1

Our goal in this section is to explain how to deduce Theorem 5.0.1 from Theorem 6.0.1 instead of Proposition 5.2.2. We consider this a more conceptual proof of Theorem 5.0.1 as Proposition 5.2.2, while elementary, uses clever cocycle manipulations at its core.

Proof of Theorem 5.0.1 using Theorem 6.0.1. We first assume that $S = \mathrm{Spec}(A)$ is affine. In this case, it suffices to show that a class $\alpha \in H^n(X, \mathcal{O}_X)$ with $n > 0$ can be killed by finite surjective maps $\pi : Y \to X$. As the map f is proper, we know that $H^n(X, \mathcal{O}_X)$ is a finite $A\{X^p\}$-module, where $A\{X^p\}$ is the non-commutative polynomial ring over A with one generator X^p satisfying the commutation relation $X^p r = r^p X^p$. This implies that there exists some monic additive polynomial $g(X^p) \in A\{X^p\}$ annihilating α. As g is additive, we have a short exact sequence

$$0 \to \ker(g) \to \mathbf{G}_a \to \mathbf{G}_a \to 0$$

of fppf sheaves of abelian groups on X. Since g is a monic polynomial, the sheaf $\ker(g)$ is representable by a finite flat commutative group scheme. As $g(\alpha) = 0$, we see that α lies in the image of $H^n(X, \ker(g)) \to H^n(X, \mathbf{G}_a) \simeq H^n(X, \mathcal{O}_X)$, i.e., α comes from a cohomology class α' with coefficients in a finite flat commutative group scheme. By Theorem 6.0.1, there exists a finite surjective map $\pi : Y \to X$ such that $\pi^*(\alpha') = 0$. By the functoriality of the constructions, it follows that π^* kills α. When S is no longer affine, we reduce to affine case treated above as in §5.2. □

6.5 An example of a torsor not killed by finite covers

Theorem 6.0.2 allows us to construct proper covers annihilating cohomology classes with coefficients in an abelian scheme. Our goal in this section is to construct an example indicating why "proper" cannot be replaced by "finite" in the preceding section. As a bonus, we get an example illustrating the necessity of strong regularity assumptions on the base scheme in Theorem 6.3.8.

6.5.1 Construction

Fix an algebraically closed field k of characteristic 0, and an elliptic curve $(E, 0)$ over k. We will construct a scheme X essentially of finite type over k satisfying the following:

1. X is a semilocal, normal, connected, 2-dimensional affine scheme with two closed points x and y. Let $U = X - \{x, y\}$ be the twice-punctured spectrum; let $X_x = \mathrm{Spec}(\mathcal{O}_{X,x})$ and $X_y = \mathrm{Spec}(\mathcal{O}_{X,y})$ be the corresponding local rings; let $U_x = U \times_X X_x$ and $U_y = U \times_X X_y$ denote the corresponding punctured spectra; and let $\widehat{U_x} = U_x \times_{X_x} \widehat{X_x}$ and $\widehat{U_y} = U_y \times_{X_y} \widehat{X_y}$ denote the punctured spectra of the corresponding completions.

2. All maps $X_x \to E$ induce the trivial map $\pi_1(\widehat{U_x}) \to \pi_1(E)$.

3. All maps $X_y \to E$ induce the trivial map $\pi_1(\widehat{U_y}) \to \pi_1(E)$.

4. There exists a map $f : U \to E$ inducing surjective maps $\pi_1(\widehat{U_x}) \to \pi_1(E)$ and $\pi_1(\widehat{U_y}) \to \pi_1(E)$ simultaneously.

We first explain the idea of the construction informally. The cone S considered in Example 6.3.7 had the property that sections of E on an open subscheme do not extend to the entire scheme. Glueing two such cones away from the cone point gives an E-torsor of infinite order on a normal scheme that does not die on passage to finite covers. The base scheme, however, is not separated. To achieve separatedness, instead of glueing naively, we look at a finite étale quadratic cover of S. The resulting scheme bares enough formal similarities with the non-separated example (namely, exactly two closed points, each of which looks like the cone S) to make this construction work. The details follow; we advise the reader willing to take the existence of X on faith to proceed to §6.5.2.

Let $\overline{S}$ the affine cone on E (considered in Example 6.3.7 with different notation), and let (S, s) denote its local scheme at the origin. Note that S is a normal, Gorenstein, local 2-dimensional scheme essentially of finite type over k. By Noether normalisation, we can pick a finite map $a : S \to \mathrm{Spec}(A)$ where A is the local ring at the origin of $\mathbf{A}^2_k$. Since S is Cohen-Macaulay, the map f is finite flat. In fact, we can even arrange for f to be totally ramified at the origin: if E is represented by the homogeneous form $y^2z = x^3 + Axz^2 + Bz^3$, then we simply choose a to be the map given by the functions y and z. Let $b : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be a finite étale cover of degree 2 with B connected. We define X to be the fibre product via

$$\begin{array}{ccc} X := S \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(B) \\ \downarrow & & \downarrow b \\ S & \xrightarrow{\quad a \quad} & \mathrm{Spec}(A). \end{array}$$

The scheme X is connected since b is étale at the origin while a is totally ramified. Moreover, being finite étale over S forces X to be a semilocal, normal, connected, 2-dimensional affine scheme with two closed points x and y. Note that since $X \to S$ is finite étale of degree 2, it is necessarily Galois. Let X_x, X_y, U_x, etc. be as above, and let's verify the desired properties.

First, we verify properties (2) and (3). The map $\widehat{U_x} \to E$ induced by a map $X_x \to E$ factors through the induced map $\widehat{X_x} \to E$ by definition. Since $\widehat{X_x}$ is a complete noetherian local scheme with algebraically closed residue field, its fundamental group vanishes, and hence the desired claim follows for X_x. We argue exactly the same way for X_y.

For property (4), we first explain how to construct f. Consider the punctured spectrum $S - \{s\}$. As explained in Examples 4.2.4 and 6.3.7, the scheme $S - \{s\}$ can be realised as the complement of the 0-section in the Zariski localisation along the 0-section of the total space of the line bundle $\mathcal{O}(-1)|_E \to E$. In particular, there exists a natural map $f_0 : S - \{s\} \to E$. Let f denote the composition $U \to S - \{s\} \to E$. Note that f is invariant under the Galois group of $X \to S$.

We will now verify that f has the desired properties. Note that since $S - \{s\}$ is ind-open in $\mathcal{O}(-1)|_E$ and both schemes are normal, the induced map $\pi_1(S - \{s\}) \to \pi_1(\mathcal{O}(-1)|_E)$ is surjective. Since we are working in characteristic 0, by homotopy invariance of the fundamental group for normal schemes, we can identify $\pi_1(\mathcal{O}(-1)|_E) \simeq \pi_1(E)$ via the natural projection. In particular, f_0 induces a surjective map $\pi_1(S - \{s\}) \to \pi_1(E)$. Moreover, the same calcuations also work after completion. Hence, the induced map $\widehat{f_0} : \widehat{S - \{s\}} \to E$ also induces a surjective map on fundamental groups. To pass to X, note that $X \to S$ is finite étale with no residue extension at the closed points. Hence, the induced map $\widehat{X_x} \to \widehat{S}$ is an isomorphism, and similarly for y. This allows us to identify $\widehat{U_x}$ with $\widehat{S - \{s\}}$ via the natural map, and similarly for y. The desired surjectivity now follows from what we already checked for S.

6.5.2 Verification

Let X be the scheme constructed in §6.5.1. Let $V_x = X - \{y\}$ and $V_y = X - \{y\}$ denote the open subschemes of X defined as the complement of each of the two closed points. Since x and y are the only two closed points of X, the pair $\{V_x, V_y\}$ defines a Zariski open cover of X with intersection $U = V_x \cap V_y$. Consider the associated Mayer-Vietoris sequence

$$\cdots H^0(V_x, E) \oplus H^0(V_y, E) \xrightarrow{\beta} H^0(U, E) \xrightarrow{\delta} H^1(X, E) \to \cdots \tag{6.1}$$

By assumption, we may pick a map $f : U \to E$ inducing surjective maps $\pi_1(\widehat{U_x}) \to \pi_1(E)$ and $\pi_1(\widehat{U_y}) \to \pi_1(E)$. Viewing f as an element of $H^0(U, E)$, we define $\alpha = \delta(f)$. We claim:

Claim 6.5.1. *The class α is not torsion. Moreover, for every finite cover $f : Y \to X$, the pullback $f^*\alpha$ is also not torsion. In particular, α does not die in a finite cover.*

Proof. Assuming that α is not torsion, the existence of norms (see Proposition 6.5.2) implies the rest. Thus, it suffices to verify that $\alpha = \delta(f)$ is not torsion. By exact sequence (6.1), it suffices to show that $n \cdot f \notin \mathrm{im}(\beta)$ for an integer $n \neq 0$. The class $n \cdot f$ is represented by the map $[n] \circ f : U \to E$ where $[n] : E \to E$ is the multiplication by n map on E. The assumption on f implies that $[n] \circ f$ induces a map $\pi_1(\widehat{U_x}) \to \pi_1(E)$ whose image has index n. On the other hand, our assumptions on X also imply that any map lying in the image of β induces trivial map $\pi_1(\widehat{U_x}) \to \pi_1(E)$. Since $\pi_1(E)$ is torsion free and non-zero, the claim follows. ☐

Lastly, we explain why the functors $H^i(-, A)$ admit "norm maps" when A is a group algebraic space, and why this forces cohomology classes killed by finite covers to be torsion. These maps were used above.

Proposition 6.5.2. *Let A be any abelian fppf sheaf on the category of all k-schemes with k a field of characteristic 0. Assume that A is represented by an algebraic space. For every finite surjective morphism $f : T \to S$ of integral normal schemes, there exist norm maps $f_* : A(T) \to A(S)$ satisfying the conditions listed in [SV96, Definition 4.1]. Moreover, the kernel of $H^i(S, A) \to H^i(T, A)$ is necessarily torsion.*

Proposition 6.5.2 is well-known, but we were unable to find a satisfactory reference. Hence, we include the sketch of a proof.

Sketch of proof. We first explain the construction of norms. Assume that $T \to S$ induces a Galois extension of function fields with group G with cardinality n. By normality of S, we identify $T/G \simeq S$. Given a T-point $a \in A(T)$, we obtain a natural map $T \to \mathrm{Map}(G, A) \simeq A^n$ given by $t \mapsto (g \mapsto a(g(t)))$. The group $S_n = S_{\#G}$ acts on $\mathrm{Map}(G, A)$, and the preceding map $T \to \mathrm{Map}(G, A)$ is equivariant for the natural embedding $G \to S_n$ given by left translation. Taking quotients as algebraic spaces, we arrive at a map $b : S \simeq T/G \to A^n/S_n = \mathrm{Sym}^n(A)$. The n-fold multiplication map $A^n \to A$ is an S_n-equivariant map to an algebraic space. Hence, it factors as $A^n \to \mathrm{Sym}^n(A) \to A$. Composing the second map with b, we obtain a map $S \to A$ that we declare to be the norm $f_*(a)$ of a. In the case $f : T \to S$ does not induce a Galois extension of function fields, one works in a Galois closure and then descend; we omit the details as they do not mattter in the sequel. One can check that this constructions verifies the conditions required on the norm map in [SV96, Definition 4.1].

For the last claim, note that a formal consequence of having norms is that for any finite surjective morphism $f : T \to S$ as above, there exists a pushforward $H^i(f_*) : H^i(T, A) \to H^i(S, A)$ such that $H^i(f_*) \circ H^i(f^*) = n$. In particular, a non-torsion class in $H^i(S, A)$ will not die on passage to finite covers if S is normal. ☐

Chapter 7

Relation to existing work

Our goal in this chapter is to relate Condition 1.0.2 to certain other existing notions of rationality in positive characteristic. The most important results are Theorems 7.1.4 and 7.2.5. The former asserts that Condition 1.0.2 implies F-rationality, and that the converse holds in the Gorenstein case; the latter asserts that Condition 1.0.2 implies pseudorationality. We also discuss the not-so-tight connection with Frobenius-splittings in §7.3.

7.1 F-rationality

We first review the definition of F-rationality, and then compare it to Condition 1.0.2. All rings occuring in this section contain $\mathbf{F}_p$ and are assumed to be F-finite, i.e., the absolute Frobenius map on the ring is assumed to be a finite morphism.

7.1.1 Review of F-rational rings

F-rationality was a characteristic p analogue of the condition of rational singularities introduced by Fedder and Watanabe in [FW89]. Originally defined in terms of tight closure, this notion was studied by Karen Smith in [Smi94] and [Smi97b]. One of her main results was a local cohomological characterisation of F-rationality which we, slightly idiosyncratically, take as our definition.

Definition 7.1.1. A noetherian excellent local $\mathbf{F}_p$-algebra $(R, \mathfrak{m})$ of dimension d is *F-rational* if it is Cohen-Macaulay, normal, and has the property that $H^d_{\mathfrak{m}}(R)$ has no proper Frobenius-stable submodules.

Remark 7.1.2. We briefly remind the reader of the Frobenius action on local cohomology. For a local $\mathbf{F}_p$-algebra $(R, \mathfrak{m})$, if $F : \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ denotes the absolute Frobenius map, then there is a natural pullback map $F^* : \mathrm{R}\Gamma_{\mathfrak{m}}(R) \to \mathrm{R}\Gamma_{\mathfrak{m}}(F_*R) \simeq F_*\mathrm{R}\Gamma_{\mathfrak{m}}(R)$. This map can be viewed as defining a p-linear endomorphism, called the Frobenius action, of $\mathrm{R}\Gamma_{\mathfrak{m}}(R)$. Under local duality (and the assumption that R is noetherian and admits a dualising complex), this action is dual to the natural trace map $\mathrm{Tr}_F : F_*\omega_R^\bullet \to \omega_R^\bullet$. In particular, the condition that $H^d_{\mathfrak{m}}(R)$ admits no proper non-zero Frobenius-stable submodules is equivalent to the same conditon for the dualising sheaf $\mathcal{H}^{-d}(\omega_R^\bullet) \simeq \omega_R$.

By work of Smith [Smi97b], Hara [Har98], and Mehta-Srinivas [MS97], a complex variety has rational singularities if and only if its reductions to positive characteristic p (suitably defined by taking spreads) have F-rational singularities for almost all primes p. In particular, this gives a lot of examples of F-rational rings.

We do not develop the theory of these rings in detail here, preferring to refer the reader to the afore mentioned references. However, we do record a useful fact concerning the behaviour of F-rationality with respect to localisation.

Proposition 7.1.3. *Let $(R, \mathfrak{m})$ be an noetherian local $\mathbf{F}_p$-algebra. Assume that R admits a dualising complex and is F-rational. Then the localisations $R_\mathfrak{p}$ are F-rational for all prime ideals $\mathfrak{p} \subset R$.*

Proof. The properties of being Cohen-Macaulay and normal localise. Thus, thanks to Remark 7.1.2, we simply need to verify that for each prime ideal $\mathfrak{p} \subset R$, the corresponding dualising module $\omega_{R_\mathfrak{p}}$ has no proper non-zero Frobenius stable submodules. The formula $\omega_{R_\mathfrak{p}} \simeq (\omega_R)_\mathfrak{p}$ and the torsion freeness of dualising sheaves allows us to view ω_R as a submodule of $\omega_{R_\mathfrak{p}}$ in a manner compatabible with Frobenius. In particular, any Frobenius stable submodule $N \subset \omega_{R_\mathfrak{p}}$ defines a Frobenius stable submodule $M = N \cap \omega_R \subset \omega_R$. Since $\omega_R \to \omega_{R_\mathfrak{p}}$ is a localisation, the same is true for $M \to N$. In particular, if $N \subset \omega_{R_\mathfrak{p}}$ is proper and non-zero, the same is true for $M \subset \omega_R$, contradicting the F-rationality of R. □

7.1.2 Relation to F-rationality

In this subsection, we prove the following theorem relating F-rationality to Condition 1.0.2.

Theorem 7.1.4. *Let $(R, \mathfrak{m})$ be a noetherian excellent local $\mathbf{F}_p$-algebra admitting a dualising complex. If R satisfies Condition 1.0.2, then R is F-rational. If R is Gorenstein, then the converse is also true.*

Proof. Both F-rationality and Condition 1.0.2 can be detected after completion. Thus, we may assume that $(R, \mathfrak{m})$ is a complete noetherian local ring.

Assume that R satisfies Condition 1.0.2. We know by Remark 3.1.6 and Corollary 5.4.3 that R is normal and Cohen-Macaulay. To show that R is F-rational, we use [Smi97b, Theorem 2.6] and [Smi94, Theorem 5.4]. Together, these theorems imply that it is enough to check that for all ideals I generated by a system of parameters, we have $IS \cap R = I$ for all finite extensions $R \to S$. This follows trivially from the definitions: Condition 1.0.2 implies that $IS = I \oplus Q$, where $Q \cap R = 0$.

For the converse direction, let R be an F-rational Gorenstein local ring of dimension $d > 0$. Given a finite extension $f : R \to S$, we need to verify that $\mathrm{ev}_f : \mathrm{Hom}(S, R) \to R$ is surjective. By the Gorenstein assumption, we can identify this map with $\mathrm{Hom}(S, \omega_R) \to \omega_R$. The image of this last map is a Frobenius-stable submodule $M \subset \omega_R$. Moreover, since the formation of M commutes with localisation, we know that M is generically non-zero. By Remark 7.1.2 and the definition of F-rationality, it follows that $M = \omega_R$ as desired. □

Remark 7.1.5. We do not know if F-rational rings satisfy Condition 1.0.2 without a Gorenstein hypothesis.

7.2 Pseudorationality

We first review the definition of pseudorationality, and thn compare it to Condition 1.0.2.

7.2.1 Review of pseudorationality

Pseudorationality was an older attempt at defining characteristic p analogue of rational singularities than F-rationality. It was defined by Lipman and Tessier in [LT81] in terms of certain good properties enjoyed by rational singularities with respect to arbitrary modifications.

Definition 7.2.1. A noetherian excellent local ring $(R, \mathfrak{m})$ of dimension d is *pseudorational* if it is normal, Cohen-Macaulay, and has the following property: for all proper birational maps $f : X \to \mathrm{Spec}(R)$, the induced map $H^d_{\mathfrak{m}}(R) \to H^d_Z(X, \mathcal{O}_X)$ is injective, where Z is the inverse image of the closed point.

Remark 7.2.2. We remind the reader of the definition of cohomology with supports: if $Z \subset X$ is a closed subset with complement U, then $H^i_Z(X, \mathcal{O}_X)$ is defined as the cohomology of a (homotopy) kernel:

$$\mathrm{R}\Gamma_Z(X, \mathcal{O}_X) = \ker(\mathrm{R}\Gamma(X, \mathcal{O}_X) \to \mathrm{R}\Gamma(U, \mathcal{O}_U)).$$

In the situation considered in Definition 7.2.1, one formally deduces that

$$\mathrm{R}\Gamma_Z(X, \mathcal{O}_X) \simeq \mathrm{R}\Gamma_{\mathfrak{m}}(\mathrm{R}f_* \mathcal{O}_X)$$

For applications, it is often convenient to work with the following dual formulation of pseudorationality, also due to Lipman-Tessier [LT81]:

Theorem 7.2.3 (Lipman-Tessier). *Let $(R, \mathfrak{m})$ be a normal, Cohen-Macaulay, excellent noetherian local ring. Then S is pseudorational if and only if for all proper birational maps $f : X \to S$, the trace map induces an isomorphism $f_* \omega_X \simeq \omega_S$.*

Proof. We may assume that S is connected. Fix a proper birational map $f : X \to S$. The Leray spectral sequence tells us that $H^0(S, f_* \omega_X) = H^0(X, \omega_X)$. As dualising sheaves are torsion free and rank 1, it follows that trace induces an isomorphism $f_* \omega_X \simeq \omega_S$ if and only if trace induces a surjection $H^0(X, \omega_X) \to H^0(S, \omega_S)$. Identifying $H^0(X, \omega_X)$ with $H^{-d}(X, \omega_X^\bullet)$, we see that trace induces a surjection if and only if

$$\mathrm{R}f_* \omega_X^\bullet \to \omega_S^\bullet$$

induces a surjection on $\mathcal{H}^{-d}$. By local duality, this last surjectivity is equivalent to the injectivity on $\mathcal{H}^d$ of the map

$$\mathrm{R}\Gamma_{\mathfrak{m}}(\mathcal{O}_S) \to \mathrm{R}\Gamma_{\mathfrak{m}}(\mathrm{R}f_* \mathcal{O}_X).$$

Since $\mathrm{R}\Gamma_{\mathfrak{m}}(\mathrm{R}f_* \mathcal{O}_X) \simeq \mathrm{R}\Gamma_Z(X, \mathcal{O}_X)$ where $Z = f^{-1}(\{\mathfrak{m}\})$, the claim follows. □

The preceding formulation of pseudorationality is useful in comparing it to other notions. For example:

Proposition 7.2.4. *Let $(R, \mathfrak{m})$ be a normal, Cohen-Macaulay, excellent noetherian local* **Q***-algebra. Then R is pseudorational if and only if $S = \mathrm{Spec}(R)$ has rational singularities.*

Proof. For the forward direction, let $f : X \to S$ be a resolution of singularities. By Theorem 7.2.3, pseduorationality of S implies that $f_* \omega_X \simeq \omega_S$ via the trace map. The Grauert-Riemenschneider vanishing theorem says that $\mathrm{R}^i f_* \omega_X = 0$ for $i > 0$. Thus, we see that the trace map induces an isomorphism $\mathrm{R}f_* \omega_X \simeq \omega_S$. As S and X are both Cohen-Macaulay, their dualising sheaves coincide with the dualising complexes (up to a shift). Hence, we find a canonical isomorphism

$$\mathrm{R}f_* \omega_X^\bullet \simeq \omega_S^\bullet.$$

Applying $\mathrm{R}\mathcal{H}\mathrm{om}(-, \omega_S^\bullet)$ and using Grothendieck duality gives us

$$\mathrm{R}f_* \mathcal{O}_X \simeq \mathrm{R}f_* \mathrm{R}\mathcal{H}\mathrm{om}(\omega_X^\bullet, \omega_X^\bullet) \simeq \mathrm{R}\mathcal{H}\mathrm{om}(\mathrm{R}f_* \omega_X^\bullet, \omega_S^\bullet) \simeq \mathrm{R}\mathcal{H}\mathrm{om}(\omega_S^\bullet, \omega_S^\bullet) \simeq \mathcal{O}_S.$$

Thus, S has rational singularities.

Conversely, assume that S has rational singularities. To show pseudorationality, by resolution of singularities, it suffices to work with resolutions $f : X \to S$. In this case, the preceding argument is reversible. $\square$

7.2.2 Relation to pseudorationality

In this subsection, we show that Condition 1.0.2 is at least as strong as pseudorationality. Of course, it is known by work of Karen Smith (see [Smi97b]) that even F-rational rings are pseudorational. Hence, the theorem below follows by combining her theorem with Theorem 7.1.4. We give a direct proof; our argument is inspired by that of Kovács [Kov00] as explained in Theorem 4.1.3.

Theorem 7.2.5. *Let* $(R, \mathfrak{m})$ *be a noetherian excellent local* $\mathbf{F}_p$*-algebra satisfying Condition 1.0.2. Then* R *is pseudorational.*

Proof. Let $S = \mathrm{Spec}(R)$. It suffices to verify the conditions mentioned in Theorem 7.2.3. The normality of R follows from Remark 3.1.6; the excellence of R implies that it is analytically unramified; the Cohen-Macaulayness of R follows from Corollary 5.4.3. Thus, it suffices to verify that for every proper birational map $f : X \to S$, the trace map $H^0(X, \omega_X) \to H^0(S, \omega_S)$ is surjective. By Theorem 5.0.2, we know that the map $\mathcal{O}_S \to \mathrm{R}f_*\mathcal{O}_S$ is split in $\mathrm{D}(\mathrm{Coh}(S))$. The choice of a splitting defines the following diagram in $\mathrm{D}(\mathrm{Coh}(S))$:

$$\mathcal{O}_S \longrightarrow \mathrm{R}f_*\mathcal{O}_S \longrightarrow \mathcal{O}_S$$

with the composite map the identity. Let $\omega_X^\bullet$ denotes the dualising complex of X normalised as usual: $\omega_X^\bullet \in \mathrm{D}^{[-d,0]}(\mathrm{Coh}(X))$ with $\mathcal{H}^{-d}(\omega_X^\bullet) \simeq \omega_X$ where ω_X is the (usual) dualising sheaf on X, and $d = \dim(X)$. Let $\omega_S^\bullet$ be the analogous object on S. Applying $\mathrm{R}\mathcal{H}\mathrm{om}(-, \omega_S^\bullet)$ to the preceding diagram and using Grothendieck duality, we obtain a diagram in $\mathrm{D}(\mathrm{Coh}(S))$ of the form

$$\omega_S^\bullet \xleftarrow{\mathrm{Tr}_f} \mathrm{R}f_*\omega_X^\bullet \longleftarrow \omega_S^\bullet$$

where Tr_f is the trace map, and the composite map is the identity. Note that because of our conventions, we have a natural identification

$$\mathcal{H}^{-d}(\mathrm{R}f_*\omega_X^\bullet) \simeq \mathcal{H}^{-d}(\mathrm{R}f_*\omega_X[d]) \simeq f_*\omega_X.$$

Applying $\mathcal{H}^{-d}$ to the preceding diagram therefore gives us a diagram in $\mathrm{Coh}(S)$ of the form

$$\omega_S \xleftarrow{\mathrm{Tr}_f} f_*\omega_X \longleftarrow \omega_S$$

where Tr_f is the corresponding trace map. Applying $H^0(S, -)$ and noting that $H^0(S, f_*\omega_X) = H^0(X, \omega_X)$ (by the Leray spectral sequence for f) now implies the desired surjectivity. $\square$

Remark 7.2.6. The proof given above actually shows that if S satisfies Condition 1.0.2 and $f : X \to S$ is any *alteration*, then the trace map $H^0(X, \omega_X) \to H^0(S, \omega_S)$ is surjective.

7.3 Frobenius splitting

The purpose of this section is review the notion of a "Frobenius splitting" in the sense of [MR85], and explain its relation to Condition 1.0.2. The basic definition is:

Definition 7.3.1. An $\mathbf{F}_p$-scheme S is *Frobenius-split* if the map $\mathcal{O}_S \to F_*\mathcal{O}_S$ induced by the absolute Frobenius map $F : S \to S$ is split in $\mathrm{Coh}(S)$.

We do not review the general theory of Frobenius-split varieties here, preferring to refer the reader to the original paper [MR85] or more recent books such as [BK05]. We will restrict ourselves to pointing out that there are strong parallels between the study of Frobenius-split varieties and the study of schemes satisfying Condition 1.0.2: both conditions impose a certain kind of positivity constraint when applied to projective varieties, neither condition is local on the scheme, and both conditions end up defining singularities in positive characteristic that are closely related to singularities classically studied over $\mathbf{C}$ using resolutions (for example, it is expected that the Frobenius-split singularities over $\mathbf{F}_p$ are expected to be analogues of the log canonical singularities).

Our main result with regards to Frobenius-splitting is the following trivial observation:

Proposition 7.3.2. *If S is a noetherian $\mathbf{F}_p$-scheme satisfying Condition 1.0.2, then S is Frobenius split.*

Proof. This implication holds by definition when the Frobenius map $F : S \to S$ is a finite morphism. In the general case, we use Proposition 3.1.4. To apply this proposition, we need to verify that $F_*\mathcal{O}_S$ is a filtered colimit of coherent $\mathcal{O}_S$-algebras. Since S is noetherian, we may write $F_*\mathcal{O}_S = \mathrm{colim}\,\mathcal{F}_i$, where $\mathcal{F}_i$ is a coherent $\mathcal{O}_S$-subsheaf of $F_*\mathcal{O}_S$. The algebra generated by each $\mathcal{F}_i$ within $F_*\mathcal{O}_S$ is then a coherent algebra (any local section f of $F_*\mathcal{O}_S$ satisifes an equation of the form $X^p - g$, where g is a local section of $\mathcal{O}_S$), and the union of all these subalgebras is $F_*\mathcal{O}_S$, as desired. $\square$

We finish with an example illustrating that the converse to Proposition 7.3.2 fails in a rather strong way. This example also illustrates the subtle arithmetic nature of a Frobenius-splitting.

Example 7.3.3. Let A be an ordinary abelian variety over an algebraically closed field k of positive characteristic. We will show that A is Frobenus-split; note that Example 3.2.7 shows that ordinary abelian varieties do not satisfy Condition 1.0.2. Since k is algebraically closed, we can (and do) identify the absolute Frobenius on A with its Frobenius map relative to k. To see that A is Frobenius-split, using Lemma 5.6.1, it suffices to show that $H^n(A, \omega_A) \to H^n(A, F^*(\omega_A))$ is injective, where $n = \dim(A)$, ω_A is the dualising line bundle on A, and $F : A \to A$ is the absolute Frobenius morphism. Since A is an abelian variety, the dualising line bundle is trivial. Hence, it suffices to show that the natural map $F^*(H^n) : H^n(A, \mathcal{O}_A) \to H^n(A, \mathcal{O}_A)$ is injective. The cohomology group $H^n(A, \mathcal{O}_A)$ is identified with $\det(H^1(A, \mathcal{O}_A))$, functorially in A. Thus, it suffices to show that $F^*(H^1) : H^1(A, \mathcal{O}_A) \to H^1(A, \mathcal{O}_A)$ is an isomorphism; this claim follows from the definition of ordinarity.

Chapter 8

Mixed characteristic

Fix a prime number p. In this chapter, we study mixed characteristic analogues of the positive characteristic theorems proven in Chapter 5. Our main goal is to prove the following theorem on the cohomology of proper morphisms, which may be considered as a mixed characteristic lift of Corollary 5.2.4:

Theorem 8.0.1. *Let $f : X \to S$ be a proper morphism of schemes with S affine and excellent. Then there exists an alteration $\pi : Y \to X$ such $\pi^*(H^i(X, \mathcal{O})) \subset p(H^i(X, \mathcal{O}))$.*

A theorem of this type is interesting only when p is not invertible on S. In the case $p = 0$ on S, Theorem 8.0.1 says that alterations kill the higher relative cohomology of the structure sheaf for proper maps — a weaker version of Proposition 5.2.2. The techniques used in Chapter 5 depend heavily on the use of Frobenius and, consequently, do not transfer over to the mixed characteristic world. Our proof of Theorem 8.0.1 is geometric in nature and, in fact, can be used to give a new proof of Theorem 5.0.1.

Our plan for this chapter is as follows. Theorem 8.0.1 is proven in §8.1: we discuss a reduction to relative dimension 0 in §8.1.1, and then prove this case in §8.1.2. In §8.2, we explain how to deduce the apparently stronger sounding Theorem 5.0.1 from Theorem 8.0.1.

8.1 The main theorem

To obtain an idea of the techniques involved in our proof of Theorem 8.0.1, consider the special case where f has relative dimension 1. If S were reduced to a point, then a natural way to proceed is as follows: replace X with its normalisation, identify the group $H^1(X, \mathcal{O})$ with the tangent space to the Picard variety $\mathrm{Pic}^0(X)$ at the origin, and prove that there exist maps of curves such that the pullback on Picard varieties is divisible by p, at least at the expense of extending the ground field. For a non-trivial family of curves, the preceding argument can be applied to solve the problem over the generic point. Using the existence of compact moduli spaces of stable curves (or, even better, stable *maps*), and some basic properties of the category of semiabelian schemes over normal schemes, we can extend the generic solution to one over an alteration of S. This is not quite enough as the alteration is no longer affine, but it does show that to prove the theorem for morphisms of relative dimension 1 it suffices to deal with the case of relative dimension 0.

In the general case, de Jong's theorems show that an arbitrary proper morphism f of relative dimension d can be altered into a sequence of d iterated curve fibrations over an alteration of the base. Thus, granting that the preceding argument works for curve fibrations, we inductively reduce the general problem to that for proper morphisms of relative dimension 0, i.e., alterations. For this last case, we carefully fibre S itself by curves while preserving certain cohomological properties and work by induction on $\dim(S)$.

In order to flesh out the preceding outline, we make the following definition:

Definition 8.1.1. Given a scheme S, we say that Condition $\mathcal{C}_d(S)$ is satisfied if S is excellent, and the following is satisfied by each irreducible component S_i of S: given a proper surjective morphism $f : X \to S_i$ of relative dimenson d with X integral, there exists an alteration $\pi : Y \to X$ such that, with $g = f \circ \pi$, we have $\pi^*(\mathrm{R}^i f_* \mathcal{O}_X) \subset p(\mathrm{R}^i g_* \mathcal{O}_Y)$ for $i > 0$.

Proving Theorem 8.0.1 can be reformulated as verifying Condition $\mathcal{C}_d(S)$ for all excellent schemes S. This verification is carried out in the sequel. More precisely, in §8.1.1, we will show that the validity of $\mathcal{C}_0(S)$ for all excellent base schemes S implies the same for $\mathcal{C}_d(S)$ for all integers d and all excellent schemes S. We then proceed to verify Condition $\mathcal{C}_0(S)$ in §8.1.2.

8.1.1 Reduction to the case of relative dimension 0

The objective of the present section is to show the relative dimension of maps considered in Theorem 8.0.1 can be brought down to 0 using suitable curve fibrations. The necessary technical help is provided by the following result, essentially borrowed from [dJ97], on extending maps between semistable curves.

Proposition 8.1.2. *Fix an integral excellent base scheme B with generic point η. Assume we have semistable curves $\phi : C \to B$ and $\phi'_\eta : C'_\eta \to \eta$, and a B-morphism $\pi_\eta : C'_\eta \to C$. If C'_η is geometrically irreducible, then we can alter B to extend π_η to a map of semistable cures over B, i.e., there exists an alteration $\tilde{B} \to B$ such that $C'_\eta \times_B \tilde{B}$ extends to a semistable curve over $\tilde{C}' \to \tilde{B}$ with $\tilde{C}'$ integral, and the map $\pi_\eta \times_B \tilde{B}$ extends to a $\tilde{B}$-map $\tilde{\pi} : \tilde{C}' \to C \times_B \tilde{B}$.*

Proof. We may extend C'_η to a proper B-scheme using the Nagata compactification theorem (see [Con07, Theorem 4.1]). By taking the closure of the graph of the rational map defined from this compactification to C by π_η, we obtain a proper dominant morphism $\phi' : C' \to B$ of integral schemes whose generic fibre is the geometrically irreducible curve $\phi'_\eta : C'_\eta \to B$, and a B-map $\pi : C' \to C$ extending $\pi_\eta : C'_\eta \to C$. The idea, borrowed from [dJ96, §4.18], is the following: modify B to make the strict transform of $C' \to B$ flat, alter the result to get enough sections which make the resulting datum generically a stable curve, use compactness of the moduli space of stable curves to extend the generically stable curve to a stable curve after further alteration, and then use stability and flatness to get a well-defined morphism from the resulting stable curve to the original one extending the existing one over the generic point. Instead of rewriting the details here, we refer the reader to [dJ97, Theorem 5.9] which directly applies to ϕ' to essentially finish the proof; the only thing to check is the integrality of $\tilde{C}'$ which follows from the irreducibility of the generic fibre $C'_\eta \times_B \tilde{B}$. □

Remark 8.1.3. Proposition 8.1.2, while sufficient for the application we have in mind, is woefully inadequate in terms of the permissible generality. Johan de Jong's techniques can, in fact, be used to show something much better: for any flat projective morphism $X \to B$, there exists an ind-proper algebraic stack $\overline{\mathcal{M}}_g(X) \to B$ parametrising B-families of stable maps from genus g curves to X

In addition to constructing maps of semistable curves, we will also need to construct maps that preserve sections. The following lemma says we can do so at a level of generality sufficient for our purposes.

Lemma 8.1.4. *Fix an integral excellent base scheme B, two semistable curves $\phi_1 : C_1 \to B$ and $\phi_2 : C_2 \to B$, and a surjective B-map $\pi : C_2 \to C_1$. Then any section of ϕ_1 extends to a section of ϕ_2 after an alteration of B, i.e., given a section $s : B \to C_1$, there exists an alteration $b : \tilde{B} \to B$ such that the induced map $\tilde{B} \to B \to C_1$ factors through a map $\tilde{B} \to C_2$.*

Proof. Let η be the generic point of B, let $s : B \to C_1$ be the section of ϕ_1 under consideration, and let $s_\eta : \eta \to C_1$ denote the restriction of s to the generic point. By the surjectivity of π, the map $\pi_\eta : (C_2)_\eta \to (C_1)_\eta$ is surjective. Thus, there exists a finite surjective morphism $\eta' \to \eta$ such that the induced map $\eta' \to C_1$ factors through some map $s'_\eta : \eta' \to C_2$. If B' denotes the normalisation of B in $\eta' \to \eta$, then the map s'_η spreads out to give a rational map $B' \dashrightarrow C_2$. Taking the closure of the graph of this rational map (over B) gives an alteration $b : \tilde{B} \to B$ such that the induced map $\tilde{B} \to C_1$ factors through a map $\tilde{s_2} : \tilde{B} \to C_2$, proving the claim. □

Proposition 8.1.2 allows us to construct maps of semistable curves by constructing them generically. We now show how to construct the desired maps generically; the idea of this construction belongs to class field theory.

Lemma 8.1.5. *Let X be a proper curve over a field k. Then there exists a field extension k' of k, a proper smooth curve Y over k' with geometrically irreducible components, and a finite flat map $\pi : Y \to X_{k'}$ such that the induced map $\pi^* : \mathrm{Pic}(X_{k'}) \to \mathrm{Pic}(Y)$ of sheaves of abelian groups is divisible by p in $\mathrm{Hom}(\mathrm{Pic}(X), \mathrm{Pic}(Y))$.*

Proof. The statement to be proven is stable under taking finite covers of X, and can be proven one connected component at a time. Thus, after picking a suitable finite extension k' of k and normalising $X_{k'}$, we may assume that X is a smooth projective geometrically connected curve of genus ≥ 1 with a rational point $x_0 \in X(k)$. The point x defines the Abel-Jacobi map $X \to \mathrm{Pic}^0(X) \subset \mathrm{Pic}(X)$ via $x \mapsto \mathcal{O}([x]) \otimes \mathcal{O}(-[x_0])$. Riemann-Roch implies that this map is a closed immersion. We set $\pi : Y \to X$ to be the normalised inverse image of X under the multiplication by p map $[p] : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$. It follows that the pullback $\pi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ factors through multiplication by p on $\mathrm{Pic}(X)$ and is therefore divisible by p. □

Lemma 8.1.5 allows us to construct covers of semistable curves that generically induce a map divisible by p on cohomology. We now show how to globalise this construction; this forms one of the primary ingredients of our proof of Theorem 8.0.1.

Proposition 8.1.6. *Let $\phi : X \to T$ be a projective family of semistable curves with T integral and excellent. Then there exists a diagram*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle \tilde{\phi}} & & \downarrow{\scriptstyle \phi} \\ \tilde{T} & \xrightarrow{\psi} & T \end{array}$$

satisfying the following:

1. *The scheme $\tilde{T}$ is integral, and the map ψ is an alteration.*

2. *$\tilde{\phi}$ is a projective family of semistable curves, and the map π is proper and surjective.*

3. *The pullback map $\psi^* \mathrm{R}^1\phi_*\mathcal{O}_X \to \mathrm{R}^1\tilde{\phi}_*\mathcal{O}_{\tilde{X}}$ is divisible by p in $\mathrm{Hom}(\psi^*\mathrm{R}^1\phi_*\mathcal{O}_X, \mathrm{R}^1\tilde{\phi}_*\mathcal{O}_{\tilde{X}})$.*

Proof. For any family $\phi : X \to T$ of projective semistable curves, there is a natural identification of $\mathrm{R}^1\phi_*\mathcal{O}_X$ with the normal bundle of the zero section of the semiabelian scheme $\mathrm{Pic}^0(X/T) \to T$. Moreover, given another semistable curve $\tilde{\phi} : \tilde{X} \to T$ and a morphism of semistable curves $\pi : \tilde{X} \to X$ over T, the induced map $\mathrm{R}^1(\pi) : \mathrm{R}^1\phi_*\mathcal{O}_X \to \mathrm{R}^1\tilde{\phi}_*\mathcal{O}_{\tilde{X}}$ can be identified as the map on the corresponding normal bundles at 0 induced by the natural morphism $\mathrm{Pic}^0(\pi) : \mathrm{Pic}^0(X/T) \to \mathrm{Pic}^0(\tilde{X}/T)$. As multiplication by n on smooth commutative group schemes induces multiplication by n on the normal bundles at 0, it follows that if $\mathrm{Pic}^0(\pi)$ is divisible by p, so is $\mathrm{R}^1(\pi)$. Given that the formation of $\mathrm{R}^1\phi_*\mathcal{O}_X$ commutes with arbitrary base change on T, it now suffices to show the following: there exists an alteration $\psi : \tilde{T} \to T$ and a morphism of semistable curves $\pi : \tilde{X} \to X \times_T \tilde{T}$ over $\tilde{T}$ such that the induced map $\mathrm{Pic}^0(\pi)$ is divisible by p. Our strategy will be to construction a solution to this problem generically on T, and then use Proposition 8.1.2 and some elementary properties of semiabelian schemes to globalise.

Let η denote the generic point of T. By Lemma 8.1.5, we can find a finite extension $\eta' \to \eta$, and a proper smooth curve $Y_{\eta'} \to \eta'$ with geometrically irreducible components such that the induced map $\mathrm{Pic}^0(X_{\eta'}) \to \mathrm{Pic}^0(Y_{\eta'})$ is divisible by p. After replacing the map $X \to T$ with its base change along the normalisation of T in $\eta' \to \eta$, we may assume that $\eta' = \eta$. The situation so far is summarised in the diagram

$$\begin{array}{ccc} \sqcup_i Y_{\eta_i} = Y_\eta & \longrightarrow & X \\ \downarrow & & \downarrow \\ \eta & \longrightarrow & T \end{array}$$

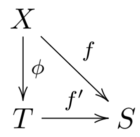
where the Y_{η_i} are the (necessarily) geometrically irreducible components of Y_η. As each of the Y_{η_i} is smooth as well, we may apply Proposition 8.1.2 to extend each Y_{η_i} to a semistable curve $Y_i \to T_i$ where $T_i \to T$ is some alteration of T, such that the map $Y_{\eta_i} \to X$ extends to a map $Y_i \to X$. Setting $\tilde{T}$ to be a dominating irreducible component of the fibre product of all the T_i over T, and setting $\tilde{X}$ to be the disjoint of $Y_i \times_{T_i} \tilde{T}$, we find the following: an alteration $\tilde{T} \to T$, a semistable curve $\tilde{X} \to \tilde{T}$ extending $Y_\eta \times_T \tilde{T}$, and a map $\tilde{\pi} : \tilde{X} \to X$ extending the existing one over the generic point. We will now check the required divisibility.

As explained earlier, it suffices to show that the resulting map $\mathrm{Pic}^0(X \times_T \tilde{T}/\tilde{T}) \to \mathrm{Pic}^0(\tilde{X}/\tilde{T})$ is divisible by p. This divisibility holds generically on $\tilde{T}$ by construction. To extend this divisibility to all of $\tilde{T}$, note that the group schemes occuring as Pic^0 of a semistable curve are all semiabelian. The normality of $\tilde{T}$ implies that restriction to the generic point is a fully faithful functor from the category of semiabelian schemes over $\tilde{T}$ to the analogous category over its generic point (see [FC90, Chapter I, Proposition 2.7]). In particular, the generic divisibility by p ensures the global divisibility by p, proving the existence of $\tilde{X}$ with the desired properties. □

Recall that we set out to reduce Theorem 8.0.1 to verifying Condition $\mathcal{C}_0(S)$. Proposition 8.1.6 allows us to make the cohomology of the fibres of a curve fibration divisible by p on passage to alterations, while de Jong's theorems allow us to alter an arbitrary proper dominant morphism into a sequence of curve fibrations over an alteration of the base. We can combine these two ingredients to make the promised reduction in relative dimension.

Proposition 8.1.7. *Let S be an excellent scheme such that Condition $\mathcal{C}_0(S)$ is satisfied. Then $\mathcal{C}_d(S)$ is satisfied for all $d \geq 0$.*

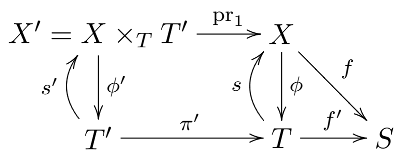
Proof. As Condition $\mathcal{C}_d(S)$ is defined in terms of the irreducible components of S, we may assume that S is integral itself. Fix integers $d, i > 0$, an integral scheme X, and a proper surjective morphism $f : X \to S$ of relative dimension d. By Corollary 5.10 of [dJ97], after replacing X by an alteration, we may assume that $f : X \to S$ factors as follows:



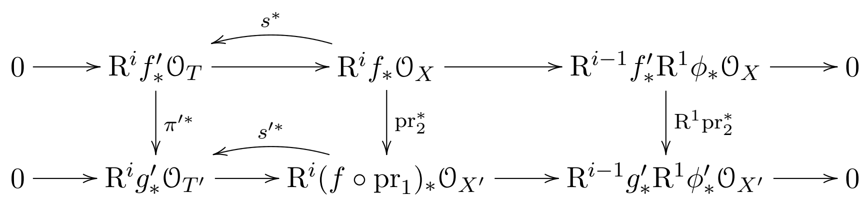
Here ϕ is a projective semistable curve, and f' is a proper surjective morphism of integral excellent schemes of relative dimension $d-1$. Also, at the expense of altering T further, we may assume that ϕ has a section $s : T \to X$. The fact that ϕ is a semistable curve gives us the formula $\mathcal{O}_T \simeq \phi_*\mathcal{O}_X$. Using the section s and the Leray spectral sequence, we find an exact sequence

$$0 \to \mathrm{R}^i f'_*\mathcal{O}_T \to \mathrm{R}^i f_*\mathcal{O}_X \to \mathrm{R}^{i-1} f'_*\mathrm{R}^1\phi_*\mathcal{O}_X \to 0$$

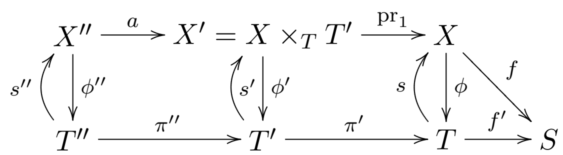
that is naturally split by the section s. Our strategy will be to prove divisibility for $\mathrm{R}^i f_*\mathcal{O}_X$ by working with the two edge pieces occuring in the exact sequence above. In more detail, we apply the inductive hypothesis to choose an alteration $\pi' : T' \to T$ such that, with $g' = f' \circ \pi'$, we have $\pi'^*(\mathrm{R}^i f'_*\mathcal{O}_T) \subset p(\mathrm{R}^i g'_*\mathcal{O}_{T'})$. The base change of ϕ and s along π' define for us a diagram



The commutativity of the preceding diagram gives rise to a morphism of exact sequences



compatible with the exhibited splittings. The map ϕ' is a semistable curve with a section s'. Applying Proposition 8.1.6 and using Lemma 8.1.4, we can find a commutative diagram



where π'' is an alteration, ϕ'' is a semistable curve, a is an alteration, s'' is a section of ϕ'' (compatible with s' and s thanks to the commutativity of the picture), such that $a^*\mathrm{R}^1\phi'_*\mathcal{O}_{X'} \to \mathrm{R}^1\phi''_*\mathcal{O}_{X''}$ is divisible by p. Setting $g'' = g' \circ \pi''$, it follows that we can make a diagram of exact sequences

$$
\begin{array}{ccccccccc}
 & & & \xleftarrow{\;s^*\;} & & & & & \\
0 & \longrightarrow & \mathrm{R}^i f'_* \mathcal{O}_T & \longrightarrow & \mathrm{R}^i f_* \mathcal{O}_X & \longrightarrow & \mathrm{R}^{i-1} f'_* \mathrm{R}^1 \phi_* \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi'^*} & \xleftarrow{\;s'^*\;} & \downarrow{\scriptstyle \mathrm{pr}_2^*} & & \downarrow{\scriptstyle \mathrm{R}^1 \mathrm{pr}_2^*} & & \\
0 & \longrightarrow & \mathrm{R}^i g'_* \mathcal{O}_{T'} & \longrightarrow & \mathrm{R}^i (f \circ \mathrm{pr}_1)_* \mathcal{O}_{X'} & \longrightarrow & \mathrm{R}^{i-1} g'_* \mathrm{R}^1 \phi'_* \mathcal{O}_{X'} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi''^*} & \xleftarrow{\;s''^*\;} & \downarrow{\scriptstyle a^*} & & \downarrow{\scriptstyle \mathrm{R}^1 a^*} & & \\
0 & \longrightarrow & \mathrm{R}^i g''_* \mathcal{O}_{T''} & \longrightarrow & \mathrm{R}^i (f \circ \mathrm{pr}_1 \circ a)_* \mathcal{O}_{X''} & \longrightarrow & \mathrm{R}^{i-1} g''_* \mathrm{R}^1 \phi''_* \mathcal{O}_{X''} & \longrightarrow & 0
\end{array}
$$

which is compatible with the exhibited splittings of each sequence. As $\mathrm{R}^1 a^*$ is divisible by p, it follows that the image of right vertical composition is divisible by p. The image of the left vertical composition is divisible by p by construction of π'. By compatibility of the morphism of exact sequences with the exhibitted splittings, it follows that the image of the middle vertical composition is also divisible by p. Replacing X'' be an irreducible component dominating X then proves the claim. □

8.1.2 The case of relative dimension 0

In this section we will verify Condition $\mathcal{C}_0(S)$ for all excellent schemes S. After unwrapping the definitions and making some preliminary reductions, one reduces to showing the following: given an alteration $f : X \to S$ with S affine and a cohomology class $\alpha \in H^i(X, \mathcal{O})$, there exists an alteration $\pi : Y \to X$ such that $p \mid \pi^*(\alpha)$. If α arose as the pullback of a class under a morphism $X \to \overline{X}$ with $\overline{X}$ proper over an affine base of dimension $\dim(S) - 1$, then we may conclude by induction using Proposition 8.1.7. The proof below will show that, at the expense of certain technical but manageable modifications, this method can be pushed through. Our main result is:

Proposition 8.1.8. *The Condition $\mathcal{C}_0(S)$ is satisfied by all excellent schemes S.*

Our proof of Proposition 8.1.8 will consist of a series of reductions which massage S until it becomes a geometrically accessible object (see Lemma 8.1.15 for the final outcome of these "easy" reductions).

Warning 8.1.9. For conceptual clarity, we often commit the following abuse of mathematics in the sequel: when proving a statement of the form that $\mathcal{C}_d(S)$ is satisfied for all integers d and a particular scheme S, we ignore the restrictions on integrality and relative dimension imposed by Condition $\mathcal{C}_d(S)$ while making certain constructions; the reader can check that in each case the statement to be proven follows from our constructions by taking suitable irreducible components (see Lemma 8.1.10 for an example). We strongly believe that this abuse, while easily fixable, enhances readability.

We first observe that the problem is Zariski local on S.

Lemma 8.1.10. *The Condition $\mathcal{C}_d(S)$ is local on S for the Zariski topology, i.e., if $\{U_i \hookrightarrow X\}$ is a Zariski open cover of X, then $\mathcal{C}_d(S)$ is satisfied if and only if $\mathcal{C}_d(U_i)$ is satisfied for all i.*

Proof. We will first show that $\mathcal{C}_d(S)$ implies $\mathcal{C}_d(U)$ for any open $j : U \to S$. By Nagata compactification (see [Con07, Theorem 4.1]), given any alteration $f : X \to U$, we can find an alteration $\overline{f} : \overline{X} \to S$ extending f over U. As $j : U \to S$ is flat, we have that $j^* \mathrm{R}^i \overline{f}_* \mathcal{O}_{\overline{X}} = \mathrm{R}^i f_* \mathcal{O}_X$. By assumption, we can find an alteration $\overline{\pi} : \overline{Y} \to \overline{X}$ such that, with $\overline{g} = \overline{f} \circ \overline{\pi}$, we have

$\overline{\pi}^*\mathrm{R}^i\overline{f}_*\mathcal{O}_{\overline{X}} \subset p(\mathrm{R}^i\overline{g}_*\mathcal{O}_{\overline{Y}})$. Restricting to U and using flat base change for $\overline{g}$ produces the desired result.

Conversely, assume there exists a cover $\{U_i \hookrightarrow X\}$ such that $\mathcal{C}_d(U_i)$ is true. Given an alteration $f : X \to S$, define $f_i : X_{U_i} \to U_i$ to be the natural map. The assumption implies that we can find alterations $\pi_i : Y_i \to X_{U_i}$ such that, with $g_i = f_i \circ \pi_i$, we have $\pi_i^*(\mathrm{R}^j f_{i*}\mathcal{O}_{X_{U_i}}) \subset p(\mathrm{R}^j g_{i*}\mathcal{O}_{Y_i})$ for each i. By Proposition 5.2.1, we can find $\pi : Y \to X$ such that $\pi \times_S U_i$ factors through π_i. As taking higher pushforwards commutes with restricting to open subsets, we see that $\pi^*(\mathrm{R}^j f_*\mathcal{O}_X) \subset \mathrm{R}^j g_*\mathcal{O}_Y$ is a subsheaf that is locally inside $p(\mathrm{R}^j g_*\mathcal{O}_Y)$. As containments between two subsheaves of a given sheaf can be detected locally, the claim follows. □

Next, we show how to localise for the topology of finite covers.

Lemma 8.1.11. *The Condition $\mathcal{C}_d(S)$ is local on S for the topology of finite covers, i.e., if $g : S' \to S$ is finite surjective, then $\mathcal{C}_d(S)$ is satisfied if and only if $\mathcal{C}_d(S')$ is satisfied.*

Proof. For the forward direction, we use the fact that a proper surjective map $f : X \to S'$ defines a proper surjective map $g \circ f : X \to S$ of the same relative dimension. For the converse direction, we use that given a proper surjective map $f : Y \to S$, the base change $Y \times_S S' \to Y$ is a finite surjective map with the additional property that $Y \times_S S'$ admits a proper surjective map to S' of the same relative dimension as f. We omit the details. □

Combining the preceding observations, we show how to pass to quasifinite covers.

Lemma 8.1.12. *If $g : S' \to S$ is quasifinite and surjective and $\mathcal{C}_d(S)$ is satisfied, then $\mathcal{C}_d(S')$ is satisfied.*

Proof. By Zariski's main theorem (Théorème 8.12.6 of [Gro66]), we can factor g as $S' \xrightarrow{j} \overline{S'} \xrightarrow{\overline{g}} S$ with j an open immersion, and $\overline{g}$ finite surjective. Lemma 8.1.11 implies that $\mathcal{C}_d(\overline{S'})$ is satisfied. The first half of the proof of Lemma 8.1.10 then shows that $\mathcal{C}_d(S')$ is also satisfied, as desired. □

Finally, we show how to étale localise:

Lemma 8.1.13. *The Condition $\mathcal{C}_d(S)$ is étale local on S, i.e., if $g : S' \to S$ is a surjective étale morphism, then $\mathcal{C}_d(S)$ is satisfied if and only if $\mathcal{C}_d(S')$ is satisfied.*

Proof. If $\mathcal{C}_d(S)$ is satisfied, then $\mathcal{C}_d(S')$ is also satisfied by Lemma 8.1.12. For the converse direction, using Lemma 8.1.10, we may assume that S and S' are both local schemes. By Lemma 6.1.2, we can find a diagram

$$\begin{array}{ccc} \sqcup_i U_i & \longrightarrow & T \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \pi} \\ S' & \xrightarrow{g} & S \end{array}$$

such that π is finite surjective, $\sqcup U_i \to T$ forms a Zariski cover, and h is some map. The commutativity of the diagram forces h to be quasifinite, while the locality of S' forces h to be surjective. Since we are assuming that $\mathcal{C}_d(S')$ is satisfied, Lemma 8.1.12 now implies that $\mathcal{C}_d(\sqcup_i U_i)$ is satisfied. Using Lemma 8.1.10, we deduce that $\mathcal{C}_d(T)$ is satisfied. Lemma 8.1.11 then allows us to conclude that $\mathcal{C}_d(S)$ is satisfied, as desired. □

Having étale localised, we prove an approximation result.

Lemma 8.1.14. *The Condition $\mathcal{C}_d(S)$ is satisfied by all affine excellent schemes S if it is satisfied by all affine schemes S of finite type over* **Z**.

Proof. By Proposition 8.1.7, we may restrict ourselves to $d = 0$ (the resulting simplification is purely notational). Assume that $\mathcal{C}_d(S)$ is satisfied whenever S is of finite type over $\mathbf{Z}$. Let S' be an arbitrary excellent affine scheme, and let $f : X' \to S'$ be a proper morphism. Given a cohomology class $\alpha' \in H^i(X', \mathcal{O}_{X'})$, the quadruple (X', S', f', α') is defined by a finite amount of algebraic data. Consequently, it can be obtained from a similar quadruple (X, S, f, α) along a base change $S' \to S$ where S has finite type over $\mathbf{Z}$. By assumption, there exists an alteration $\pi : Y \to X$ such that $\pi^*\alpha$ is divisible by p. It follows then that a dominating irreducible component of $\pi' : Y \times_S S' \to X'$ provides the desired alteration. □

Next, we complete at p.

Lemma 8.1.15. *The Condition $\mathcal{C}_d(S)$ is satisfied by all affine excellent schemes S if it is satisfied by all affine schemes S of finite type over the Witt vector ring $\mathrm{W}(\overline{\mathbf{F}}_p)$.*

Proof. We first explain the idea informally. Using Lemmas 8.1.14 and 8.1.13, we easily reduce to verifying $\mathcal{C}_0(S)$ for S of finite type over the strict henselisation R^{sh}, where $R = \mathbf{Z}_{(p)}$ is the localisation of $\mathbf{Z}$ at p. To deduce the statement over R^{sh} from that over $\widehat{R^{\mathrm{sh}}} = \mathrm{W}(\overline{\mathbf{F}}_p)$, we use Popescu's approximation theorem which allows us to write $\widehat{R^{\mathrm{sh}}}$ is an inductive limit of smooth R^{sh}-algebras. The crucial point here is that any smooth R^{sh}-algebra with non-empty special fibre has an R^{sh}-valued point; the details of this argument now follow.

By Proposition 8.1.7 and Lemma 8.1.14, it suffices to show that $\mathcal{C}_0(S)$ is satisfied whenever S is affine and of finite type over $\mathbf{Z}$. As there is nothing to show when $p \in \mathcal{O}_S^*$, Nagata compactification (see [Con07, Theorem 4.1]) easily reduces us to verifying $\mathcal{C}_0(S)$ for S affine and of finite type over $R = \mathbf{Z}_{(p)}$, the localisation of $\mathbf{Z}$ at p. By Lemma 8.1.13 and a limit argument, we reduce to verifying $\mathcal{C}_0(S)$ for S affine and of finite type over R^{sh}. Let S be such a scheme, and let $f : X \to S$ be an alteration of S. As the groups $H^i(X, \mathcal{O}_X)$ are finite $\mathcal{O}_S$-modules, in order to make the groups $H^i(X, \mathcal{O})$ divisible by p, it suffices to work one class at a time. Let $\alpha \in H^i(X, \mathcal{O}_X)$ with $i > 0$ be a cohomology class of degree $i > 0$. By assumption, we know there exists an alteration $\widehat{\pi} : \widehat{Y} \to X_{\widehat{R^{\mathrm{sh}}}}$ such that $p|\widehat{\pi}^*(\alpha)$, where $\widehat{R^{\mathrm{sh}}} \simeq \mathrm{W}(\overline{\mathbf{F}}_p)$ is the p-adic completion of R^{sh}. By the main theorem of [Pop85], we know that the completion $R^{\mathrm{sh}} \to \widehat{R^{\mathrm{sh}}}$ is an ind-smooth morphism, i.e., we can write $\widehat{R^{\mathrm{sh}}} = \operatorname{colim} R_i$, where $R^{\mathrm{sh}} \to R_i$ is finite type and smooth. Furthermore, since $R^{\mathrm{sh}} \to \widehat{R^{\mathrm{sh}}}$ has a non-empty special fibre, the same is true for $R^{\mathrm{sh}} \to R_i$ for each i. By virtue of everything being of finite presentation, there exists an index i and an alteration $\pi_i : Y_i \to X_{R_i}$ giving $\widehat{\pi}$ over $\widehat{R^{\mathrm{sh}}}$ such that $p|\pi_i^*(\alpha)$. As $R^{\mathrm{sh}} \to R_i$ is smooth with non-empty special fibre, we can cut R_i with appropriately chosen hyperplane sections to find a quotient $R_i \to T$ such that the composite $R^{\mathrm{sh}} \to R_i \to T$ is finite étale. Since R^{sh} is strictly henselian and T is local, it follows that $R^{\mathrm{sh}} \simeq T$. The fibre $\pi_i \times_{R_i} T$ is then easily seen to do the job. □

We have reduced the proof of Theorem 8.0.1 to showing Condition $\mathcal{C}_0(S)$ for affine schemes S of finite type over $B = \mathrm{Spec}(\mathrm{W}(\overline{\mathbf{F}}_p))$. Given an alteration of such an S, the non-quasi-finite locus of the alteration is a closed subset $Z \subset S$ of codimension ≥ 2 that is often called the *center* of the alteration. Our strategy for proving Theorem 8.0.1 is to construct, at the expense of localising a little on S, a partial compactification $S \hookrightarrow \overline{S}$ with $\overline{S}$ proper over a lower dimensional base such that Z remains closed in $\overline{S}$. This last condition ensures that the alteration in question can be extended to an alteration of $\overline{S}$ without changing the center. As the center has not changed, the cohomology of the newly created alteration maps onto that of the older alteration, thereby paving the way for an inductive argument via Proposition 8.1.7. The precise properties needed to carry out the above argument are ensured by the presentation lemma that follows.

Lemma 8.1.16. *Let $B = \mathrm{Spec}(\mathrm{W}(\overline{\mathbf{F}}_p))$ be the maximal unramified extension of $\mathbf{Z}_p$. Let $\hat{S}$ be a local integral scheme, flat and essentially of finite type B with relative dimension ≥ 1. Assume that the residue field of $\hat{S}$ at the closed point has positive characteristic. Given a closed subset $\hat{Z} \subset \hat{S}$ of codimension ≥ 2, we can find a diagram of B-schemes of the form*

$$\begin{array}{ccccccc} s \in Z & \xrightarrow{i_Z} & S & \xrightarrow{j} & \overline{S} & \xleftarrow{i} & \partial\overline{S} \\ & \searrow & & \searrow & \downarrow{\scriptstyle \pi} & \swarrow & \\ & & & & W & & \end{array}$$

satisfying the following:

1. *All the schemes in the diagram above are of finite type over B.*
2. *S is an integral scheme, i_Z is a closed subscheme, s is a closed point, and the germ of i_Z at s agrees with $\hat{Z} \subset \hat{S}$.*
3. *i is the inclusion of a Cartier divisor, and j is the open dense complement of i.*
4. *W is an integral affine scheme with $\dim(W) = \dim(S) - 1$.*
5. *π is proper, $\pi \mid_S$ is affine, and both these maps have fibres of equidimension 1.*
6. *$\pi \mid_Z$ and $\pi \mid_{\partial\overline{S}}$ are finite. In particular, $j(i_Z(Z))$ is closed in $\overline{S}$.*

Proof. We begin by choosing an ad hoc finite type model of $\hat{Z} \hookrightarrow \hat{S}$ over B, i.e., we find a map $i_Y : Y \to T$ and a point $y \in Y$ satisfying the following: the map i_Y is a closed immersion of finite type integral B-schemes with codimension ≥ 2, and the germ of i_Y at y is the given map $\hat{Z} \hookrightarrow \hat{S}$. Next, we choose an ad hoc compactification $T \hookrightarrow \overline{T}$ over B, i.e., $\overline{T}$ is a projective B-scheme containing T as a dense open subscheme. By replacing T with the complement of a suitable ample divisor missing the point y in the special fibre (and hence in all of $\overline{T}$ by properness), we may assume that the complement $\partial\overline{T}$ is an ample divisor flat over B. We denote by $\overline{Y}$ the closure of Y in $\overline{T}$, and by $\partial\overline{Y} = \overline{Y} - Y$ its boundary. As Y had codimension ≥ 2, its closure $\overline{Y}$ also has codimension ≥ 2, while the boundary $\partial\overline{Y}$ has codimension ≥ 3 as Y is not contained in $\partial\overline{T}$. Our goal is to modify our choices of T and $\overline{T}$ to eventually find the required S and $\overline{S}$.

Let d denote the relative dimension of $\overline{T}$ over B. By construction, this is the relative dimension of $\hat{S}$ over B as well. The next step is to find a finite morphism $\phi : \overline{T} \to \mathbf{P}^d$ such that $\phi(y) \notin \phi(\partial\overline{T})$. We find such a map by repeatedly projecting. In slightly more detail, say we have a finite morphism $\phi : \overline{T} \to \mathbf{P}^N$ for some $N > d$ such that $\phi(y) \notin \phi(\partial\overline{T})$. Then $\phi(\partial\overline{T})$ is a closed subscheme of codimension ≥ 2. Moreover, by the flatness of $\partial\overline{T}$ over B, the same is true in the special fibre $\mathbf{P}^N \times_B \overline{\mathbf{F}}_p$. By basic facts of projective geometry in the special fibre, we can find a line ℓ through $\phi(y)$ that does not meet $\phi(\partial\overline{T})$. By the ampleness of $\partial\overline{T}$, this line cannot entirely be contained in $\phi(\overline{T})$. Thus, we can find a point on it that is not contained in $\phi(\overline{T})$. By projecting from this point, we see that we can find a finite morphism $\phi : \overline{T} \to \mathbf{P}^{N-1}$ such that $\phi(y) \notin \phi(\partial\overline{T})$. So far the discussion was taking place in the special fibre. However, by choosing a lift of this point to a B-point by smoothness of $\mathbf{P}^N$ and using the properness of $\partial\overline{T}$ to transfer the non-intersection condition from the special fibre to the total space, it follows that we can make this construction over B. Continuing this way, we can find a finite morphism $\phi : \overline{T} \to \mathbf{P}^d$ with the same property. As $\phi(\partial\overline{T})$ is now a very ample Cartier divisor, its complement $U \hookrightarrow \mathbf{P}^d$ is an open affine containing $\phi(y)$. We may now replace T with $\phi^{-1}(U)$ and Y with $\overline{Y} \cap \phi^{-1}(U)$ (this does not change the closure as $y \in \overline{Y} \cap \phi^{-1}(U)$ and $\overline{Y}$ is irreducible) to assume that we have produced the following: an algebraisation $i_Y : Y \to T$ of $\hat{Z} \to \hat{S}$ for some point $y \in Y$, a compactification $T \hookrightarrow \overline{T}$, and a finite morphism $\phi : \overline{T} \to \mathbf{P}^d$ such that $T = \phi^{-1}(U)$ for some open affine $U \in \mathbf{P}^d$ that is the complement of a very ample divisor H, flat over B.

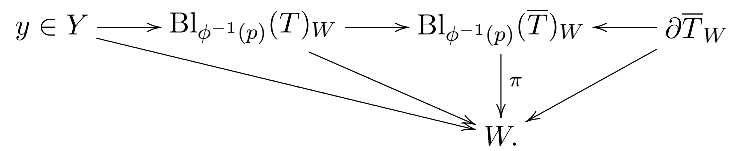
Now we project once more to obtain the desired curve fibration. As explained earlier, the closure $\overline{Y}$ has codimension ≥ 2 in $\overline{T}$. Since we do not know that it is flat over B, the most we can say is that its image $\phi(\overline{Y})$ has codimension ≥ 1 in the special fibre $\mathbf{P}^d \times_B \overline{\mathbf{F}}_p$. On the other hand, we know that $\partial\overline{T}$ was a B-flat divisor. Thus, its image $\phi(\partial\overline{T})$ also has codimension ≥ 1 in the special fibre $\mathbf{P}^d \times_B \overline{\mathbf{F}}_p$. It follows that $\phi(\overline{Y} \cup \partial\overline{T})$ has codimension ≥ 1 in the special fibre $\mathbf{P}^d \times_B \overline{\mathbf{F}}_p$. By choosing a closed point not in this image inside U and lifting to a B-point as above, we find a B-point $p : B \to U \subset \mathbf{P}^d$ whose image does not intersect $\phi(\overline{Y} \cup \partial\overline{T})$. Projecting from this point gives rise to the following diagram:

$$\begin{array}{ccccccc} \mathrm{Bl}_{\phi^{-1}(p)}(T) & \xrightarrow{a} & \mathrm{Bl}_{\phi^{-1}(p)}(\overline{T}) & \xrightarrow{b} & \mathrm{Bl}_p(\mathbf{P}^d) & \xrightarrow{c} & \mathbf{P}(T_p(\mathbf{P}^d)) \simeq \mathbf{P}^{d-1} \\ \downarrow & & \downarrow & & \downarrow & & \\ T & \longrightarrow & \overline{T} & \longrightarrow & \mathbf{P}^d. & & \end{array}$$

The horizontal maps enjoy the following properties: c is a $\mathbf{P}^1$-fibration (in the Zariski topology), b is a finite surjective morphism, and a is an open immersion. In particular, the composite map cb is a proper morphism with fibres of equidimension 1. As the map $\phi : \overline{T} \to \mathbf{P}^d$ was chosen to ensure that $\phi^{-1}(U) = T$, the composite map cba can be factored as

$$\mathrm{Bl}_{\phi^{-1}(p)}(T) \to \mathrm{Bl}_p(U) \to \mathbf{P}^{d-1}.$$

The first map in this composition is finite surjective as ϕ is so, while the second map is an affine morphism with fibres of equidimension 1 thanks to Lemma 8.1.17 below. It follows that the composite map cba is an affine morphism with fibres of equidimension 1. Lastly, by our choice of p, the map cb restricts to a finite map on $\overline{Y}$ and $\overline{T}$ (here we identify subschemes of $\overline{T}$ not intersectng $\phi^{-1}(p)$ with those of the blowup). As explained earlier, the boundary $\partial\overline{Y}$ has codimension ≥ 3 in $\overline{T}$. This implies that its special fibre has codimension ≥ 2. Therefore, its image in $\mathbf{P}^{d-1}$ has codimension ≥ 1. It follows that we can find an open affine $W \hookrightarrow \mathbf{P}^{d-1}$ not meeting the image of $\phi(\partial\overline{Y})$. Restricting the entire picture thus obtained to W, we find a diagram that looks like



Setting $s = y$, $Z = Y$, $S = \mathrm{Bl}_{\phi^{-1}(p)}(T)_W$, $\overline{S} = \mathrm{Bl}_{\phi^{-1}(p)}(\overline{T})_W$, and $\partial\overline{S} = \partial\overline{T}_W$ implies the claim. $\square$

In the proof of the preceding presentation lemma, an elementary fact concerning blowups was used. This is recorded below.

Proposition 8.1.17. *Fix an affine regular base scheme B. Let $H \hookrightarrow \mathbf{P}^n \times B$ be a divisor that is flat and relatively ample over B, and let U be the complement. For any point $p \in U(B)$, the blowup map $\mathrm{Bl}_p(U) \to \mathbf{P}(T_p(\mathbf{P}^n))$ is an affine morphism with fibres of equidimension* 1.

Proof. Let $b : \mathrm{Bl}_p(\mathbf{P}^n) \to \mathbf{P}^n$ be the blowup map, and let $\pi : \mathrm{Bl}_p(\mathbf{P}^n) \to \mathbf{P}(T_p(\mathbf{P}^n))$ be the morphism defined by projection. It is easy to see that π is a $\mathbf{P}^1$-bundle. In fact, it can be identified with the projectivisation of the rank 2 vector bundle $\mathcal{O}(1) \oplus \mathcal{O}$ on $\mathbf{P}(T_p(\mathbf{P}^n))$, with the exceptional divisor corresponding to the zero section of $\mathcal{O}(1)$. As the ample divisor H was disjoint from the center of the blowup, $b^*(H)$ defines an ample divisor on the fibres of π. By semicontinuity, it follows that for any vector bundle $\mathcal{E} \in \mathrm{Vect}(\mathrm{Bl}_p(\mathbf{P}^n))$, the higher pushforwards $R^i\pi_*\mathcal{E}(nH)$ vanish for $i > 0$ provided n is sufficiently large. By the regularity of all schemes in sight, it follows that the same is true for any coherent sheaf. As $\mathrm{Bl}_p(U) \hookrightarrow \mathrm{Bl}_p(\mathbf{P}^n)$ is the complement of $b^*(H)$, it follows by Serre's affineness criterion that $\pi|_{\mathrm{Bl}_p(U)}$ is affine. The claim about the fibre dimension follows from the observation that, geometrically, the fibre of $\pi|_{\mathrm{Bl}_p(U)}$ over a line ℓ passing through p, viewed as a point $[\ell] \in \mathbf{P}(T_p(\mathbf{P}^n))$, is simply $\ell \cap U$ which is a non-empty affine curve in ℓ thanks to the choice of p and the positivity of H □

Before proceeding to the proof of Theorem 8.0.1, we record a cohomological consequence of certain geometric hypotheses. The hypotheses in question are the kind ensured by Lemma 8.1.16, while the consequences are those used in proof of Theorem 8.0.1.

Proposition 8.1.18. *Fix a noetherian scheme $\overline{X}$ of finite Krull dimension. Let $j : X \hookrightarrow \overline{X}$ be a dense open immersion whose complement $\Delta \subset \overline{X}$ is affine and the support of a Cartier divisor. Then $H^i(\overline{X}, \mathcal{O}) \to H^i(X, \mathcal{O})$ is surjective for all $i > 0$.*

Proof. As $\Delta \subset \overline{X}$ is the support of a Cartier divisor, the complement j is an affine map. This implies that

$$j_*\mathcal{O}_X \simeq \mathrm{R}j_*\mathcal{O}_X.$$

Now consider the exact sequence

$$0 \to \mathcal{O}_{\overline{X}} \to j_*\mathcal{O}_X \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is defined to be the cokernel. As $j_*\mathcal{O}_X \simeq \mathrm{R}j_*\mathcal{O}_X$, the middle term in the preceding sequence computes $H^i(X, \mathcal{O})$. By the associated long exact sequence on cohomology, to show the claim, it suffices to show that $H^i(\overline{X}, \mathcal{Q}) = 0$ for $i > 0$. By construction, we have a presentation

$$j_*\mathcal{O}_X = \mathrm{colim}_n\, \mathcal{O}_{\overline{X}}(n\Delta).$$

Thus, we also have a presentation

$$\mathcal{Q} = \mathrm{colim}_n\, \mathcal{O}_{\overline{X}}(n\Delta)/\mathcal{O}_{\overline{X}}.$$

This presentation defines a natural increasing filtration $F^\bullet(\mathcal{Q})$ with

$$F^n(\mathcal{Q}) = \mathcal{O}_{\overline{X}}(n\Delta)/\mathcal{O}_{\overline{X}}$$

for $n \geq 0$. The associated graded pieces of this filtration are

$$\mathrm{gr}^n_F(\mathcal{Q}) = \mathcal{O}_{\overline{X}}(n\Delta) \otimes \mathcal{O}_\Delta.$$

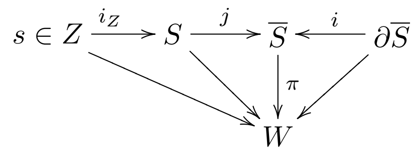
In particular, these pieces are supported on Δ which is an affine scheme by assumption. Consequently, these pieces have no higher cohomology. By a standard devissage argument, it follows that the sheaves $F^n(\mathcal{Q})$ have no higher cohomology for any n. As cohomology commutes with filtered colimits of sheaves on noetherian schemes of finite Krull dimension, it follows that $\mathcal{Q}$ has no higher cohomology either, establishing the claim. □

We now have enough tools to finish proving Theorem 8.0.1.

Proof of main theorem. Our goal is to show that Condition $\mathcal{C}_0(S)$ is satisfied by an induction on $\dim(S)$. By Lemma 8.1.15 and Lemma 8.1.10, we may assume that $\hat{S}$ is a local integral scheme that is essentially of finite type over B with a characteristic p residue field at the closed point. We give an argument below in the case that $\hat{S}$ is flat over B. The only way this flatness fails to occur is if $\hat{S}$ is an $\mathbf{F}_p$-algebra. The reader can check that all our arguments go through in this case as well, provided a few trivial modifications are made to Lemma 8.1.16. We prefer to not make these modifications here for clarity of exposition.

We assume that $\hat{S}$ is flat over B. If $\dim(S) = 1$, then any alteration of $\hat{S}$ is actually a finite morphism. In this case, as there is no relative cohomology, there is nothing to show. We may therefore assume that the relative dimension of $\hat{S}$ over B is at least 1.

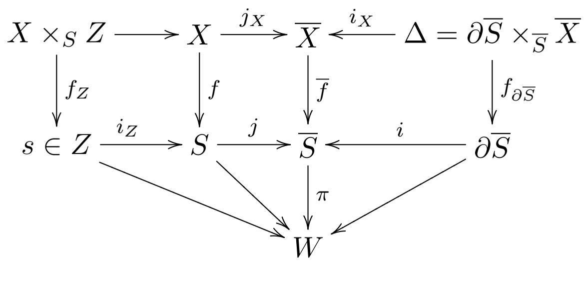
With the assumptions as above, given an alteration $\hat{f} : \hat{X} \to \hat{S}$, we want to find an alteration $\hat{g} : \hat{Y} \to \hat{X}$ such that $\hat{g}^*(H^i(\hat{X}, \mathcal{O})) \subset p(H^i(\hat{Y}, \mathcal{O}))$. As $\hat{f}$ is an alteration, the center $\hat{Z} \subset \hat{S}$ is a closed subset of codimension ≥ 2 such that $\hat{f}$ is finite away from $\hat{Z}$. Applying the conclusion of Proposition 8.1.16, we can find a diagram



satisfying the conditions guaranteed by Proposition 8.1.16. The next step is to extend the alteration $\hat{f}$ to an alteration $\overline{f} : \overline{X} \to \overline{S}$ which gives $\hat{f}$ as the germ at s and has center $Z \subset \overline{S}$. This can be accomplished as follows: normalising $\overline{S} - Z$ in the function field of $\hat{X}$ gives rise to a finite morphism $X' \to \overline{S} - Z$ which agrees with $\hat{f}$ over $\hat{S} - \hat{Z}$. Glueing X' and $\hat{X}$ along their fibres over $\hat{S} - \hat{Z}$ (and spreading out a little) defines an alteration $\overline{f}'$ of an open subset $U \subset \overline{S}$ satisfying the following:

- $\overline{S} - U \subset Z - (Z \cap \hat{S})$, and therefore, $s \in U$.
- $\overline{f}'$ agrees with $\hat{f}$ at s, and $\overline{f}'$ is finite on $U - (Z \cap U)$.

By Nagata compactification (see [Con07, Theorem 4.1]), we obtain an alteration $\overline{f} : \overline{X} \to \overline{S}$ which is finite away from Z and agrees with $\hat{f}$ over $\hat{S}$. Let $f : X \to S$ denote the restriction of $\overline{f}$ to S. We summarise the preceding constructions by the following diagram:



Here the first row is obtained by base change from the second row via $\overline{f}$. In particular, i_X is the inclusion of a Cartier divisor. As $\overline{f}$ is finite away from the closed set Z which does not meet $\partial\overline{S}$, it follows that $f_{\partial\overline{S}}$ is a finite morphism. In particular, the scheme $\partial\overline{S} \times_{\overline{S}} \overline{X}$ is affine. Applying Proposition 8.1.18 to the map i_X, we find that $H^i(\overline{X}, \mathcal{O}) \to H^i(X, \mathcal{O})$ is surjective for $i > 0$. Since

$\dim(W) < \dim(S)$, the inductive hypothesis and Proposition 8.1.7 ensure that Condition $\mathcal{C}_d(W)$ is true for all d. As $\overline{X} \to W$ is proper surjective, we can find an alteration $\overline{g} : \overline{Y} \to \overline{X}$ such that $\overline{g}^*(H^i(\overline{X}, \mathcal{O})) \subset p(H^i(\overline{Y}, \mathcal{O}))$. It follows that a similar p-divisibility statement holds for the alteration $g : Y \to X$ obtained by restricting $\overline{g}$ to $X \hookrightarrow \overline{X}$. Lastly, by flat base change, we know that $H^i(X, \mathcal{O})$ generates $H^i(\hat{X}, \mathcal{O})$ as a module over $\Gamma(\hat{S}, \mathcal{O})$. Thus, pulling back this alteration to $\hat{X} \hookrightarrow X$ produces the desired alteration $\hat{g} : \hat{Y} \to \hat{X}$. □

Remark 8.1.19. One noteworthy feature of the proof of Proposition 8.1.8 is the following: while trying to show $\mathcal{C}_0(S)$ is satisfied, we use that $\mathcal{C}_d(S')$ is satisfied for $d > 0$ and certain affine schemes S' with $\dim(S') < \dim(S)$. We are allowed to make such arguments thanks to Proposition 8.1.7 and induction. However, this phenomenon explains why Proposition 8.1.7 appears before Proposition 8.1.8 in this paper, despite the relevant statements naturally preferring the opposite order.

Remark 8.1.20. Theorem 8.0.1, while ostensibly being a statement about coherent cohomology, is actually motivic in that it admits obvious analogues for most natural cohomology theories such as de Rham cohomology or étale cohomology. In the former case, we can use Theorem 8.0.1 and the Hodge-to-de Rham spectral sequence to reduce to proving a p-divisibility statement for $H^i(X, \Omega^j_{X/S})$ with $j > 0$. Choosing local representatives for differential forms and extracting p-th roots out of the relevant functions can then be shown to solve the problem. In étale cohomology, there is an even stronger statement: for any noetherian excellent scheme X, there exist *finite* covers $\pi : Y \to X$ such that $\pi^*(H^i_{\text{ét}}(X, \mathbf{Z}_p)) \subset p(H^i_{\text{ét}}(Y, \mathbf{Z}_p))$ for any fixed $i > 0$; this statement follows from Theorem 6.0.1 using the exact sequences of (continuous p-adic) étale sheaves

$$0 \to \mathbf{Z}_p \xrightarrow{p} \mathbf{Z}_p \to \mathbf{Z}/p \to 0.$$

It seems reasonable to expect an analogous statement in crystalline cohomology holds as well.

Remark 8.1.21. The proof given above of Theorem 8.0.1 actually shows the following: given a proper morphism $f : X \to S$ with S excellent, we can find an alteration $\pi : Y \to S$ such that, with $g = \pi \circ f$, we have:

1. $\pi^*(\mathrm{R}^1 f_* \mathcal{O}_X) \subset p(\mathrm{R}^1 g_* \mathcal{O}_Y)$.
2. The map $\tau_{\geq 2} \mathrm{R} f_* \mathcal{O}_X \to \tau_{\geq 2} \mathrm{R} g_* \mathcal{O}_Y$ is divisible by p as a morphism in $\mathrm{D}(\mathrm{Coh}(S))$.

The reason one has to truncate above 2 and not 1 in the second statement above is that divisibility by p in a Hom-group imposes torsion conditions not visible when requiring individual classes to be divisible by p. For instance, the second conclusion above implies that the p-torsion in $\mathrm{R}^i f_* \mathcal{O}_X$ for $i \geq 2$ can be killed by alterations. We do not know how to prove an H^1-analogue of this statement: when S is affine, this analogue amounts to verifying that functions $H^0(X, \mathcal{O}_X/p)$ on the special fibre of X lift to the functions $H^0(X, \mathcal{O}_X)$ on all of X provided we allow passage to alterations.

Finally, we record a global corollary of Theorem 8.0.1 that was already proven above.

Corollary 8.1.22. *Let $f : X \to S$ be a proper morphism with S excellent. Then there exists an alteration $\pi : Y \to X$ such that, with $g = f \circ \pi$, we have $\pi^*(\mathrm{R}^i f_* \mathcal{O}_X) \subset p(\mathrm{R}^i g_* \mathcal{O}_Y)$ for each $i > 0$.*

Proof. One can trace through our constructions to see we have already proven this. Alternately, this follows by combining Theorem 8.0.1 and Lemma 8.1.10. □

8.2 A new proof of Theorem 5.0.1

Our goal in this section is to explain an alternative proof of Theorem 5.0.1, the main theorem of Chapter 5, using Theorem 8.0.1. Recall that Theorem 5.0.1 asserts that, in positive characteristic, the higher cohomology of proper maps can be killed by finite covers. Applying Theorem 8.0.1 in positive characteristic only allows us to kill cohomology on passage to proper covers. The point of the proof that follows, therefore, is that annihilation by proper covers implies annihilation by finite covers for coherent cohomology. We refer the reader to §6.5 for an example with étale cohomology with coefficients in an abelian variety where such an implication fails.

Proof of Theorem 5.0.1 using Theorem 8.0.1. Let $f : X \to S$ be a proper map. We first explain the idea informally. Using Corollary 8.1.22, we find a proper surjective map $Y \to X$ annihilating the cohomology of f; next, we find another proper surjective map $Y'' \to Y$ annihilating the relative cohomology of $Y \to X$; lastly, we check that the Stein factorisation of $Y'' \to X$ does the job.

In more detail, by repeatedly applying the conclusion of Corollary 8.0.1 and using Lemma 5.1.2 , we may find a proper surjective map $\pi' : Y' \to X$ such that, with $g' = f \circ \pi'$, we have that $\tau_{\geq 1}\mathrm{R}f_*\mathcal{O}_X \to \tau_{\geq 1}\mathrm{R}g_*\mathcal{O}_Y$ is trivial. Applying a similar construction this time to the map $\pi' : Y' \to X$, we find a map $\pi^\flat : Y'' \to Y'$ such that, with $\pi'' = \pi^\flat \circ \pi'$, we have that $\tau_{\geq 1}\mathrm{R}\pi'_*\mathcal{O}_Y \to \tau_{\geq 1}\mathrm{R}\pi''_*\mathcal{O}_{Y''}$ is 0. The picture obtained thus far is:

$$\begin{array}{ccccc} Y'' & \xrightarrow{\pi^\flat} & Y' & & \\ & \searrow^{\pi''} & \downarrow{\scriptstyle \pi'} & \searrow^{g'} & \\ & & X & \xrightarrow{f} & S. \end{array}$$

The diagram restricted to X gives rise to the following morphism of exact triangles in $\mathrm{D}(\mathrm{Coh}(X))$:

$$\begin{array}{ccccccc} \mathcal{O}_X & = & \mathcal{O}_X & \longrightarrow & 0 & \longrightarrow & \mathcal{O}_X[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \pi'_*\mathcal{O}_{Y'} & \longrightarrow & \mathrm{R}\pi'_*\mathcal{O}_{Y'} & \longrightarrow & \mathcal{Q}_{Y'/X} & \longrightarrow & \pi'_*\mathcal{O}_{Y'}[1] \\ \downarrow{\scriptstyle a} & \overset{s}{\swarrow} & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c=0} & & \downarrow{\scriptstyle a[1]} \\ \pi''_*\mathcal{O}_{Y''} & \longrightarrow & \mathrm{R}\pi''_*\mathcal{O}_{Y''} & \longrightarrow & \mathcal{Q}_{Y''/X} & \longrightarrow & \pi''_*\mathcal{O}_{Y''}[1]. \end{array}$$

Here the objects $\mathcal{Q}_{Y'/X}$ and $\mathcal{Q}_{Y''/X}$ are defined as the (homotopy) cokernels of the corresponding maps, the vertical arrows are the natural pullback maps, and the dotted arrow s is a chosen lifting of b guaranteed by the condition $c = 0$. Applying $\mathrm{R}f_*$ to the above diagram, we find a factorisation:

$$\begin{array}{ccc} \mathrm{R}f_*\mathcal{O}_X & \xrightarrow{\qquad h \qquad} & \mathrm{R}f_*(\pi''_*\mathcal{O}_{Y''}) \\ & {\scriptstyle d}\searrow \quad \nearrow{\scriptstyle e} & \\ & \mathrm{R}(f\circ\pi')_*\mathcal{O}_{Y'} \simeq \mathrm{R}g'_*\mathcal{O}_{Y'}. & \end{array}$$

The map d induces the 0 map on $\tau_{\geq 1}$ by construction. It follows that the same is true for the map h. On the other hand, the sheaf $\pi''_*\mathcal{O}_{Y''}$ is a coherent sheaf of algebras on X. Hence, it corresponds to a finite morphism $\pi : Y \to X$. In fact, π is simply the Stein factorisation of π''. In particular, π is surjective. It then follows that $\pi : Y \to X$ is a finite surjective morphism such that, with $g = f \circ \pi$, the induced map $\tau_{\geq 1} \mathrm{R} f_* \mathcal{O}_X \to \tau_{\geq 1} \mathrm{R} g_* \mathcal{O}_Y$ is 0, as desired. □

Remark 8.2.1. There is an alternative and more conceptual explanation of the preceding reduction from proper covers to finite covers in the case of H^1. Namely, let $\alpha \in H^1(X, \mathcal{O}_X)$ be a cohomology class, and let $f : Y \to X$ be a proper surjective map such that $f^*\alpha = 0$. We may represent α as a $\mathbf{G}_a$-torsor $T \to X$. The assumption on Y then says that there is an X-map $Y \to T$. The image Y' of Y in T is both proper over X (as Y is so) and affine over X (as T is so). Consequently, $Y' \hookrightarrow T \to X$ is a finite surjective morphism annihilating α. The key idea underlying this argument is that the universal $\mathbf{G}_a$-torsor $* \to B(\mathbf{G}_a)$ is an affine morphism. Hence, one can make this argument work in arbitrary cohomological degree provided one is willing to work with higher stacks. Indeed, then the relevant statement is simply that $* \to K(\mathbf{G}_a, n)$ is an affine morphism for all $n \geq 0$, which can be proven by induction on n.

Chapter 9

Almost direct summands

Let V be a p-adic discrete valuation ring whose residue field k satisfies $[k : k^p] < \infty$; for example, we could take V to be a finite extension of $\mathbf{Z}_p$. Our goal is to prove the following theorem:

Theorem 9.0.1. *Let R be a smooth V-algebra, and let $f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ be a finite surjective morphism with S normal. Assume that f is étale away from a simple normal crossings divisor $D \subset \mathrm{Spec}(R)$. Then $f^* : R \to S$ is a direct summand as an R-module map.*

This chapter is organised as follows. In §9.1, we go over the basics of almost ring theory, the primary ingredient of the proof of Theorem 9.0.1. We then prove the theorem in §9.2.

9.1 Review of almost ring theory

Our proof uses almost ring theory as discovered by Tate in [Tat67], and developed by Faltings in [Fal88] and [Fal02] with p-adic Hodge theoretic applications in mind. The book [GR03] provides a more systematic treatment of almost ring theory, while [Ols09] provides a detailed and comprehensible presentation of the arithmetic applications of Faltings' ideas. We review below the aspects of the theory most relevant to the proof of Theorem 9.0.1, deferring to the other sources for proofs.

9.1.1 Almost mathematics

Let $\overline{V}$ denote a valuation ring whose value group Λ is dense in $\mathbf{Q}$, and let $\mathfrak{m} \subset \overline{V}$ be the maximal ideal. Note that $\mathfrak{m}$ is necessarily not finitely generated. For each $\alpha \in \Lambda$, let $\mathfrak{m}_\alpha$ be the (necessarily principal) ideal of elements of valuation at least α, and let π denote a generator of $\mathfrak{m}_1$.

Example 9.1.1. Let V denote a finite extension of $\mathbf{Z}_p$ corresponding to a field extension $\mathbf{Q}_p \to K$. We let $\overline{V}$ denote the integral closure of V in an infinitely ramified extension L of K. Then $\overline{V}$ is a valuation ring whose value group is dense in $\mathbf{Q}$ and, consequently, almost ring theory applies. The two main examples for us will be the cases when L is a totally ramified $\mathbf{Z}_p$-extension of K, or when $L = \overline{K}$ is the algebraic closure of K.

The maximal ideal $\mathfrak{m}$ can be thought of as the ideal of elements with positive valuation. As the value group is dense, it follows that $\mathfrak{m}^2 = \mathfrak{m}$. This observation implies that the category Σ of $\mathfrak{m}$-torsion $\overline{V}$-modules is a Serre subcategory of the abelian category $\overline{V}$-Mod of $\overline{V}$-modules. We call modules in Σ *almost zero modules*. By general nonsense, we may form the quotient abelian category

$$\overline{V}^a\text{-Mod} := \overline{V}\text{-Mod}/\Sigma$$

of *almost* $\overline{V}$-modules. We denote the localisation functor by $M \mapsto M^a$. With this notation, we have the following description of maps in $\overline{V}^a$-Mod (see §2.2.4 of [GR03]):

$$\mathrm{Hom}_{\overline{V}^a}(M^a, N^a) = \mathrm{Hom}_{\overline{V}}(\mathfrak{m} \otimes_{\overline{V}} M, N)$$

Remark 9.1.2. The novelty of the theory of almost modules rests crucially on the density of $\Lambda \subset \mathbf{Q}$. If $\Lambda \subset \mathbf{Q}$ is not dense, then the category of $\mathfrak{m}$-torsion modules is not a Serre subcategory as it is not closed under extensions. In the classical case that $\Lambda \subset \mathbf{Q}$ is discrete, the Serre closure of the subcategory of $\mathfrak{m}$-torsion modules is simply the category of all torsion modules. The corresponding quotient category then is simply the category of vector spaces over the fraction field of the discrete valuation ring.

As $\Sigma \subset \overline{V}$-Mod is closed under tensor products, the quotient $\overline{V}^a$-Mod inherits the structure of an abelian $\otimes$-category with the quotient map $\overline{V}$-Mod $\to \overline{V}^a$-Mod being a monoidal functor. This formalism allows one to systematically define "almost analogs" of standard notions of ring theory and, indeed, develop "almost algebraic geometry." Informally, we may think of almost algebraic geometry as the study of algebraic geometry over $\overline{V}$ where all the results hold up to $\mathfrak{m}$-torsion. To see this program carried out in the appropriate level of generality, we suggest [GR03]. We will adopt the more pragmatic stance of explaining the notions we need to precisely state Faltings' almost purity theorem. We start with the following set of definitions, borrowed from [Ols09], which allow us to define the fundamental notion of an almost étale morphism:

Definition 9.1.3. Let A be a $\overline{V}$-algebra, and let M be an A-module. We say that

1. M is *almost projective* if $\mathrm{Ext}^i_A(M, N)$ is almost zero for all A-modules N and $i > 0$.
2. M is *almost flat* if $\mathrm{Tor}^A_i(M, N)$ is almost zero for all A-modules N and $i > 0$.
3. M is *almost faithfully flat* if it is almost flat and if for any A-modules N_1 and N_2, the natural map
$$\mathrm{Hom}_R(N_1, N_2) \to \mathrm{Hom}_R(N_1 \otimes M, N_2 \otimes M)$$
has an almost zero kernel.
4. M is *almost finitely generated* if for every $\alpha \in \Lambda_+$, there exists a finitely generated A-module N_α and a π^α-isomorphism $N_\alpha \simeq M$, i.e., there are maps $\phi_\alpha : N_\alpha \to M$ and $\psi_\alpha : M \to N_\alpha$ such that $\phi_\alpha \circ \psi_\alpha = \pi^\alpha \circ \mathrm{id}$ and $\psi_\alpha \circ \phi_\alpha = \pi^\alpha \circ \mathrm{id}$.

Remark 9.1.4. The properties defined in Definition 9.1.3 are all invariant under almost isomorphisms and, consequently, depend only on the almost isomorphism class $M^a \in \overline{V}^a$-Mod. This is clear for flatness and projectivity by the exactness of the localisation functor $\overline{V}$-Mod $\to \overline{V}^a$-Mod. The issue of finite generation is more delicate, and we refer the interested reader to §2.3 of [GR03] for a detailed discussion. We will content ourselves by pointing out that the rather artificial looking definition given above applies to say that $\mathfrak{m} \subset \overline{V}$ is an almost finitely generated $\overline{V}$-module.

Remark 9.1.5. The $\otimes$-structure on $\overline{V}^a$-Mod gives rise, by adjointness, to an internal Hom functor denoted alHom. For given $\overline{V}$-modules M and N, this functor can also be defined as

$$\mathrm{alHom}(M^a, N^a) = \mathrm{Hom}_{\overline{V}}(M, N)^a$$

Then one can show that a module M is almost projective if and only if $\mathrm{alHom}(M^a, \cdot)$ is an exact functor. Similarly, a module M is almost flat if and only if $M^a \otimes \cdot$ is an exact functor.

Remark 9.1.6. As all the properties defined in Definition 9.1.3 are invariant under almost isomorphisms, it is tempting to define notions such as almost projectivity purely in terms of the internal homological algebra of the abelian category $\overline{V}^a\text{-Mod}$, i.e., in terms of the internal Ext functors. However, this approach suffers from two serious defects. First, as the category $\overline{V}^a\text{-Mod}$ lacks enough projectives (the generating object $\overline{V}^a$ is not projective), one is forced to resort to a Yoneda definition of the Ext groups which is clumsy to work with. More seriously, as the Yoneda definition pays no attention to the $\otimes$-structure, the resulting theory does not interact well with the $\otimes$-structure.

Once we have access to a good theory of flatness and finite generation, one can copy the standard notions in algebraic geometry to arrive at the fundamental notion of an almost étale morphism.

Definition 9.1.7. A morphism $A \to B$ of $\overline{V}$-algebras is called an *almost étale covering* if

1. B is almost finitely generated, almost faithfully flat, and almost projective as an A-module.
2. B is almost finitely generated and almost projective as a $B \otimes_A B$-module.

Example 9.1.8. We discuss the first non-trivial example of an almost étale morphism as discovered by Tate in his study [Tat67] of p-divisible groups. Our exposition follows that of Faltings (see [Fal88, Theorem1.2]). Let V be a finite extension of $\mathbf{Z}_p$, and fix a tower $V = V_0 \subset V_1 \subset \dots \subset V_n \subset \dots$ of extensions (normalisations in finite extensions of the fraction field) such that $\Omega^1_{V_{n+1}/V_n}$ has V_{n+1}/p as a quotient for each n. One can produce such a tower starting with a totally ramified $\mathbf{Z}_p$-extension of V which, in turn, can be produced using local class field theory. Set $V_\infty = \mathrm{colim}_n V_n$. Then V_∞ is a valuation ring whose value group is dense in $\mathbf{Q}$ and, consequently, almost ring theory applies. One of the key ideas in Tate's work is that V_∞ behaves like the maximal extension of V unramified in characteristic 0 provided one works in the almost category. More precisely, given a finite extension $V \to W$, we set W_n to be normalisation of $W \otimes_V V_n$, and $W_\infty = \mathrm{colim}_n W_n$. Then Tate showed (in different language) that $V_\infty \to W_\infty$ is an almost étale extension (see [Tat67, §3.2, Proposition 9]). This fact can be regarded as the 1-dimensional case of Faltings' almost purity theorem discussed below (except that Faltings allows imperfect residue fields), and implies that the absolute integral closure V^+ of V is an ind-almost étale extension of V_∞. Tate uses this description to compute the Galois cohomology of the p-adic completion $\widehat{V^+}$ of V^+ (after inverting p). These calculations form the basis for most later theorems in p-adic Hodge theory.

Finally, we record a fact concerning the almost analog of finite flat morphisms that is used in the proof of Theorem 9.0.1.

Lemma 9.1.9. *Let $f : A \to B$ be an inclusion of $\overline{V}$-algebras. Assume that f makes B an almost projective and almost faithfully flat A-module. Then the cokernel $\mathrm{coker}(f)$ is an almost projective A-module.*

Proof. For any three A-modules M, N, and K, we have an isomorphism of functors

$$\mathrm{RHom}(M \otimes^{\mathrm{L}} N, K) \simeq \mathrm{RHom}(M, \mathrm{RHom}(N, K)).$$

It follows then that if M and N are almost projective, so is $M \otimes^{\mathrm{L}} N$. If, in addition, one of M or N is also almost flat, then it follows that $M \otimes N$ is almost projective. In particular, in the case at hand, $B \otimes_A B$ is almost projective. Now note that we have an exact sequence

$$0 \to A \to B \to \mathrm{coker}(f) \to 0.$$

Tensoring this over A with B gives new exact sequence

$$0 \to B \to B \otimes_A B \to \mathrm{coker}(f) \otimes_A B \to 0.$$

The multiplication map on B splits this exact sequence. Thus, $\mathrm{coker}(f) \otimes_A B$ is a direct summand of the almost projective A-module $B \otimes_A B$ and, consequently, almost projective itself. At this point we can simply invoke [GR03, Lemma 4.1.5] and be done. $\square$

9.1.2 Faltings' purity theorem

We now state the version of Faltings' almost purity theorem most relevant to Theorem 9.0.1: the case of good reduction with ramification supported along a simple normal crossings divisor. There exist more general statements in the literature, and we refer the reader to [Fal02, §2b] for the most general known statement formulated in the language of toroidal geometry.

Let V be a p-adic discrete valuation ring whose residue field k satisfies $[k : k^p] < \infty$. Let K be the fraction field of V, and let $\overline{V}$ be the normalisation of V in a fixed algebraic closure of its fraction field. We will work with almost ring theory over $\overline{V}$. Let R be a smooth V-algebra such that $R \otimes_V \overline{V}$ is a domain. Assume we are given a presentation as an étale morphism $V[T_1, \cdots, T_d] \to R$. We define

$$R_n = \overline{V}[T_i^{\frac{1}{n!}}] \otimes_{V[T_i]} R.$$

By construction, we have natural maps $R_n \to R_m$ for $n \leq m$, and we set $R_\infty = \mathrm{colim}_n R_n$. Note that $R \to R_n$ is a finite flat morphism, and consequently, $R \to R_\infty$ is ind-finite flat and, consequently, faithfully flat. It is also ind-unramified away from the divisor defined by the function $pT_1 \cdots T_d$. The purity theorem says that $R \to R_\infty$ is the maximal extension of R with this last property provided one works in the almost category, similar to the situation in Example 9.1.8.

Theorem 9.1.10 (Faltings)**.** *Let $f : R \to S$ be the normalisation of R in a finite extension of its fraction field. Assume that the induced map $f \otimes_R R[\frac{1}{pT_1 \cdots T_d}]$ is étale. If S_n denotes the normalisation of $S \otimes_R R_n$, and $S_\infty = \mathrm{colim}_n S_n$, then the induced map $R_\infty \to S_\infty$ is an almost étale covering.*

Remark 9.1.11. Theorem 9.1.10 can be thought of as a mixed characteristic analog of Abhyankar's lemma without tameness restrictions. Recall that Abhyankar's lemma says (see [Gro03, Exposé XIII, Proposition 5.2]) that for any regular local ring R, the maximal extension of $R[T_1, \cdots, T_d]$ tamely ramified along the divisor associated to $T_1 \cdots T_d$ may be obtained by adjoining all n-power roots of the T_i's, where n runs through integers invertible on R. The purity theorem does away with the tameness restrictions at the expense of only *almost* describing the maximal extension ramified along a normal crossings divisor (in mixed characteristic).

Remark 9.1.12. We will indicate how Theorem 9.1.10 is applied to p-adic Hodge theory focussing on the simplest possible case. Assume V is a finite extension of $\mathbf{Z}_p$. Let X be a smooth V-scheme. One goal of p-adic Hodge theory is to relate the p-adic étale cohomology groups $H^*_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_p)$ to the de Rham cohomology of X. Assume for simplicity that $X = \mathrm{Spec}(R)$ is affine, and $X_{\overline{K}}$ is a $K(G, 1)$, i.e., for any local system $\mathcal{F}$ on $X_{\overline{K}}$, we have an identification

$$H^i_{\text{ét}}(X_{\overline{K}}, \mathcal{F}) \simeq H^i(\pi_1(X_{\overline{K}}), \mathcal{F}(\overline{R}_{\overline{K}}))$$

where $\overline{R}$ is the maximal extension of R unramified in characteristic 0. Thus, we reduce ourselves to relating the de Rham cohomology of X to the cohomology of the group $\pi_1(X_{\overline{K}})$ of continuous automorphisms of $\overline{R}$ over $R_{\overline{V}}$. In this setting, we will sketch the following:

1. A relation between $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_p) \simeq H^i(\pi_1(X_{\overline{K}}), \mathbf{Z}_p)$ and $H^i(\pi_1(X_{\overline{K}}), \widehat{\overline{R}})$, where $\widehat{\overline{R}}$ denotes the p-adic completion.

2. A de Rham approach to calculating $H^i(\pi_1(X_{\overline{K}}), \widehat{\overline{R}})$.

The relation between $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_p) \simeq H^i(\pi_1(X_{\overline{K}}), \mathbf{Z}_p)$ and $H^i(\pi_1(X_{\overline{K}}), \widehat{\overline{R}})$: There is a natural extension of scalars map map $H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \widehat{\overline{V}} \to H^i(\pi_1(X_{\overline{K}}), \widehat{\overline{R}})$. This map turns out to be an almost isomorphism. Faltings shows this by globalising the right hand side to a cohomology theory $\mathcal{H}^*(-)$ on smooth V-schemes in such a way that the preceding map extends to a natural transformation $H^*_{\text{ét}}(-, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \widehat{\overline{V}} \to \mathcal{H}^*(-)$ of functors. Moreover, Faltings then equips the theory $\mathcal{H}^*(-)$ with Poincare duality and various other geometric structures (such as Chern classes and cup products). He then proceeds to show that aforementioned natural transformation preserves these additional geometric structures. It then formally follows that the natural transformation is an equivalence of functors, analogous to how Weil cohomology theories are uniquely determined once the coefficients are pinned down.

The computation of $H^i(\pi_1(X_{\overline{K}}), \widehat{\overline{R}})$: By mimicking the constructions above after shrinking R a little if necessary, we may produce an explicit extension $R_{\overline{V}} \to R_\infty$ (essentially by extracting p-power roots of well-chosen units) that is the maximal extension of $R_{\overline{V}}$ unramified in characteristic 0 provided one works in the almost category, i.e, there is a natural map $R_\infty \to \overline{R}$ that is an almost étale covering. The group of continuous automorphisms of $\overline{R}$ over R_∞ is a normal subgroup H of $\pi_1(X_{\overline{K}})$ by construction. We let Δ_∞ denote the quotient $\pi_1(X_{\overline{K}})/H$. We may then consider the Hochschild-Serre spectral sequence

$$H^p(\Delta_\infty, H^q(H, \widehat{\overline{R}})) \Rightarrow H^{p+q}(\pi_1(X_{\overline{K}}), \widehat{\overline{R}}).$$

As the map $R_\infty \to \overline{R}$ is ind-almost étale, we obtain an almost isomorphism $H^0(H, \widehat{\overline{R}}) \simeq H^*(H, \widehat{\overline{R}})$. Identifying the left hand side with $\widehat{R_\infty}$, we see that the above spectral sequence almost degenerates to give an almost isomorphism of algebras

$$H^*(\Delta_\infty, \widehat{R_\infty}) \simeq H^*(\pi_1(X_{\overline{K}}), \widehat{\overline{R}}).$$

As the extension $R_{\overline{V}} \to R_\infty$ is explicitly constructed by extracting p-power roots of a system of units, we may Galois equivariantly identify Δ_∞ with $\mathbf{Z}_p(1)^d$. The cohomology algebra on the left is then easily identified with an explicit exterior algebra which, in turn, may be related to de Rham cohomology of X by some further computations with differentials. The algebra on the right, as explained above, is essentially étale cohomology of $X_{\overline{K}}$. Thus, we obtain the sought-after Hodge theoretic relation.

9.2 Proof of Theorem 9.0.1

Our goal in this section is to prove Theorem 9.0.1. Correspondingly, let V be as in Theorem 9.0.1, and let $\overline{V}$ be the normalisation of V in a fixed algebraic closure of its fraction field. One of the main ideas informing the construction of almost ring theory is that the passage from algebraic geometry over V to almost algebraic geometry over $\overline{V}$ is fairly faithful. The following lemma is one manifestation of this idea, and crucial in our proof.

Lemma 9.2.1. *Let R be a flat V-algebra essentially of finite type, R_∞ a flat $\overline{V}$-algebra, and let $R \to R_\infty$ be a faithfully flat map lying over the natural map $V \to \overline{V}$. If M is an R-module such that $M \otimes_R R_\infty$ is zero in $\overline{V}^a$-*Mod*, then $M = 0$.*

Proof. We may assume that R and R_∞ are local rings, the map $R \to R_\infty$ is a local map, and that $R/pR \neq 0$. Let $x \in M$ be a non-zero element. The assumption that $M \otimes_R R_\infty$ is almost zero implies that $\mathfrak{m}R_\infty \subset \mathrm{Ann}(x \otimes 1)$. Thus, the ideal $\mathrm{Ann}(x \otimes 1)$ contains arbitrarily small p-powers. On the other hand, by the flatness of $R \to R_\infty$, we see that $\mathrm{Ann}(x \otimes 1) = \mathrm{Ann}(x) \otimes_R R_\infty$. As $\mathrm{Ann}(x)$ is an ideal in R, the smallest power of p it contains is bounded above 0 since $x \neq 0$. By faithful flatness of R_∞ over $\overline{V}$, scaling by elements of R_∞ cannot decrease the power of p. Thus, the smallest power of p occuring in $\mathrm{Ann}(x \otimes_R 1)$ is bounded above 0 as well, contradicting the earlier conclusion that $\mathfrak{m}R_\infty \subset \mathrm{Ann}(x \otimes 1)$. □

We are now in a position to prove Theorem 9.0.1.

Proof of Theorem 9.0.1. Our goal is to show that the exact sequence

$$0 \to R \to S \to \mathcal{Q} \to 0 \tag{9.1}$$

is split, where $\mathcal{Q}$ is defined to be the cokernel. The obstruction to the existence of a splitting is an element $\mathrm{ob}(f) \in \mathrm{Ext}^1_R(\mathcal{Q}, R)$. We will show this class almost vanishes after a suitable big extension, and then appeal to Lemma 9.2.1.

The assumptions in the theorem imply that there exists an étale morphism $V[T_1, \cdots, T_d] \to R$ such that $R \to S$ is étale over $R[\frac{1}{pT_1 \cdots T_d}]$. Using this presentation, we define rings R_n, S_n, R_∞, and S_∞ as in §9.1.2. In particular, $R \to R_\infty$ is ind-finite flat. The picture over R_∞ can be summarised as:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R_\infty & \longrightarrow & S \otimes_R R_\infty & \longrightarrow & \mathcal{Q} \otimes_R R_\infty & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R_\infty & \longrightarrow & S_\infty & \longrightarrow & \mathcal{Q}_\infty & \longrightarrow & 0. \end{array}$$

Here the first row is obtained by tensoring the exact sequence (9.1) with R_∞, while $\mathcal{Q}_\infty$ is the cokernel of $R_\infty \to S_\infty$. Theorem 9.1.10 implies that the map $R_\infty \to S_\infty$ is an almost étale covering. By Lemma 9.1.9, the quotient $\mathcal{Q}_\infty$ is an almost projective R_∞-module. Hence, the second exact sequence is almost zero when viewed as an element of $\mathrm{Ext}^1_{R_\infty}(\mathcal{Q}_\infty, R_\infty)$. By the commutativity of the diagram, the first exact sequence then defines an almost zero element of $\mathrm{Ext}^1_{R_\infty}(\mathcal{Q} \otimes_R R_\infty, R_\infty)$. On the other hand, by the flatness of $R \to R_\infty$, we also know that this element is simply $\mathrm{ob}(f) \otimes 1$ under the natural isomorphism $\mathrm{Ext}^1_R(\mathcal{Q}, R) \otimes_R R_\infty \simeq \mathrm{Ext}^1_{R_\infty}(\mathcal{Q} \otimes_R R_\infty, R_\infty)$. By Lemma 9.2.1 applied to the submodule of $\mathrm{Ext}^1_R(\mathcal{Q}, R)$ generated by $\mathrm{ob}(f)$, we see that $\mathrm{ob}(f) = 0$, as desired. □

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