A NON-ALGEBRAIC HOM-STACK

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Let $A_0/\mathbf{C}$ be a non-zero abelian variety. Let $S := \mathrm{Spec}(\mathbf{C}[\![t]\!])$ be the displayed trait with $S_n := \mathrm{Spec}(\mathbf{C}[t]/(t^n)) \subset S$. Let $f : X \to S$ be a map satisfying the following:

(1) f is projective and flat, and its fibres are geometrically connected curves.
(2) The fibre $X_0 := X \times_S S_0$ is an irreducible nodal plane cubic.
(3) X is a regular scheme.

Let $A = A_0 \times S$ be the constant abelian scheme over S associated to A. Then we will show:

Proposition 0.1. *The stack $\mathcal{F} := \underline{\mathrm{Hom}}_S(X, B(A))$ is not algebraic.*

Recall that $B(A) \simeq B(A_0) \times S$. This allows us to write

$$\mathcal{F}(T) := \underline{\mathrm{Hom}}_S(X, B(A))(T) := \mathrm{Hom}_T(X \times_S T, B(A) \times_S T) \simeq \mathrm{Hom}_S(X \times_S T, B(A)) \simeq \mathrm{Hom}(X \times_S T, B(A_0))$$

for any S-scheme T, i.e., the groupoid $\mathcal{F}(T)$ is the groupoid of A_0-torsors on $X \times_S T$.

Remark 0.2. Proposition 0.1 seems to contradict [Aok06b, Theorem 1.1]. The problem encountered below is the non-effectivity of formal objects for $\underline{\mathrm{Hom}}_S(X, B(A))$. The same problem is mentioned in the Erratum [Aok06a] to [Aok06b]; unfortunately, the Erratum goes on the assert that $\underline{\mathrm{Hom}}_S(X, Y)$ is algebraic if Y is separated, which also contradicts Proposition 0.1 as $Y = B(A)$ is certainly separated.

To prove Proposition 0.1, by [Sta14, Tag 07x8], it is enough to show the following:

Lemma 0.3. *The canonical map $\mathcal{F}(S) \to \lim \mathcal{F}(S_n)$ is not essentially surjective.*

Proof. Unwinding definitions, it is enough to check that $H^1(X, A_0) \to \lim H^1(X_n, A_0)$ is not surjective. As X is regular and projective, by Raynaud [Ray70, Proposition XIII.2.6], each A_0-torsor over X is projective and hence torsion. In particular, the group $H^1(X, A_0)$ is torsion. It is thus enough to show: (a) the group $H^1(X_0, A_0)$ is non-torsion, and (b) the maps $H^1(X_{n+1}, A_0) \to H^1(X_n, A_0)$ are surjective for all n. For (a), let $\pi : \mathbf{P}^1 \to X_0$ be the normalization, and assume $0 \in \mathbf{P}^1(\mathbf{C})$ and $\infty \in \mathbf{P}^1(\mathbf{C})$ lie over the node $x \in X_0$. Choose a non-torsion point $P \in A_0(\mathbf{C})$, and consider the trivial A_0-torsor $T_0' \to \mathbf{P}^1$. Using translation by P to identify the fibres over 0 and ∞ of $T_0' \to \mathbf{P}^1$, we can descend T_0' along π to obtain a non-torsion A_0-torsor $T_0 \to X_0$, proving (a). For (b), deformation theory shows that the obstruction to deforming an A_0-torsor $T_n \to X_n$ to an A_0-torsor $T_{n+1} \to X_{n+1}$ lies in $H^2(X_0, \omega)$ for a suitable vector bundle ω on X_0; the latter vanishes as X_0 is a curve, proving the claim. ☐

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