FORMAL GLUEING OF MODULE CATEGORIES

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Fix a noetherian scheme $X$, and a closed subscheme $Z$ with complement $U$. Our goal is to explain a result of Artin that describes how coherent sheaves on $X$ can be constructed (uniquely) from coherent sheaves on the formal completion of $X$ along $Z$, and those on $U$ with a suitable compatibility on the overlap. In fact, the main result is a general (i.e., non-noetherian) local version, which we will state once we have the following definition in place.

**Definition 0.1.** Given a ring $A$ and an element $f \in A$, a ring map $\phi : A \rightarrow B$ is said to be $f$-adically faithfully flat if $\phi$ is flat, and $\phi/f : A/fA \rightarrow B/fB$ is faithfully flat. The map $\phi : A \rightarrow B$ is said to be an $f$-adic neighbourhood if it is $f$-adically faithfully flat, and the induced map $A/fA \rightarrow B/fB$ is an isomorphism.

We let $\text{Mod}(A)$ denote the abelian category of $A$-modules over a ring $A$, while $\text{Mod}_{fg}(A)$ denotes the subcategory of finitely generated $A$-modules. The main result is:

**Theorem 0.2.** Let $A$ be a ring, and let $f \in A$. Let $\phi : A \rightarrow B$ be an $f$-adic neighbourhood. Then the natural map

$$\mathcal{F} : \text{Mod}(A) \rightarrow \text{Mod}(A_f) \times_{\text{Mod}(B_f)} \text{Mod}(B)$$

is an equivalence.

The category $\text{Mod}(A_f) \times_{\text{Mod}(B_f)} \text{Mod}(B)$ appearing on the right side of the expression in Theorem 0.2 is the category of triples $(M_1, M_2, \psi)$ where $M_1$ is an $A_f$-module, $M_2$ is a $B$-module, and $\psi : M_1 \otimes_{A_f} B_f \simeq M_2 \otimes_B B_f$ is a $B_f$-isomorphism. The natural map referred to in Theorem 0.2 is defined by $\mathcal{F}(M) = (M_f, M_B, \text{can})$ where $\text{can} : M_f \otimes_{A_f} B_f \simeq M_B \otimes_B B_f$ is the natural isomorphism. We generally refer to objects of this category as “glueing data.” The motivation behind this terminology is topological and explained in Remark 0.5.

A useful special case of Theorem 0.2 is when $A$ is noetherian, and $B$ is a completion of $A$ at an element $f$. The completion $A \rightarrow B$ is flat by basic theorems in noetherian ring theory, and the functor $M \mapsto M \otimes_A B$ can be identified with the $f$-adic completion functor when $M$ is finitely generated. Thus, we obtain:

**Corollary 0.3.** Let $A$ be a noetherian ring, let $f \in A$ be an element, and let $\hat{A}$ be the $f$-adic completion of $A$. Then the obvious functors (localisation and completion) define an equivalence

$$\text{Mod}_{fg}(A) \simeq \text{Mod}_{fg}(A_f) \times_{\text{Mod}_{fg}(A_f)} \text{Mod}_{fg}(\hat{A})$$

**Remark 0.4.** The equivalence of Theorem 0.2 preserves the obvious $\otimes$-structure on either side. Thus, it defines equivalences of various categories built out of the pair $(\text{Mod}(A), \otimes)$, such as the category of $A$-algebras.

**Remark 0.5.** Theorem 0.2 may be regarded as an algebraic analogue of the following trivial theorem from topology: given a manifold $X$ with a closed submanifold $Z$ having complement $U$, specifying a sheaf on $X$ is the same as specifying a sheaf on $U$, a sheaf on an unspecified tubular neighbourhood $T$ of $Z$ in $X$, and an isomorphism between the two resulting sheaves along $T \cap U$. The lack of tubular neighbourhoods in algebraic geometry forces us to work with formal neighbourhoods instead, rendering the proof a little more complicated.

**Remark 0.6.** We suspect that Theorem 0.2 follows formally from the existence of a good model structure for the flat topology. Specifically, if one has a model structure where open immersions are cofibrations, then the square

$$\begin{array}{ccc}
\text{Spec}(B_f) & \longrightarrow & \text{Spec}(B) \\
\downarrow & & \downarrow \\
\text{Spec}(A_f) & \longrightarrow & \text{Spec}(A)
\end{array}$$

will be a homotopy pushout square. Evaluating the fpqc-stack $\text{Mod}(-)$ on this pushout diagram would then allow us to deduce Theorem 0.2 from usual fpqc-descent.
1. Generalities

Fix a ring $A$ and an element $f$.

**Definition 1.1.** An $A$-module $M$ said to be an $f^\infty$-torsion $A$-module if for each $m \in M$, there exists an $n > 0$ such that $f^n m = 0$. The full subcategory of $\text{Mod}(A)$ spanned by $f^\infty$-torsion modules is denoted $\text{Mod}(A)[f^\infty]$, while the subcategory spanned by $f^n$-torsion modules is denoted $\text{Mod}(A)[f^n]$.

We first reformulate the definition of $f$-adic faithful flatness in terms of the category $\text{Mod}(A)[f^\infty]$.

**Lemma 1.2.** Fix a ring map $\phi : A \to B$. Then the following are equivalent

1. The map $\phi$ is $f$-adically flat.
2. The map $\phi$ is flat, and the map $\text{Spec}(B/fB) \to \text{Spec}(A/fA)$ is surjective.
3. The map $\phi$ is flat, and the functor $M \mapsto M \otimes_A B$ is faithful on $\text{Mod}(A)[f]$.
4. The map $\phi$ is flat, and the functor $M \mapsto M \otimes_A B$ is faithful on $\text{Mod}(A)[f^\infty]$.

**Proof.** (1) and (2) being equivalent is standard, while the equivalence of either with (3) follows by identifying $f$-torsion $A$-modules with $A/f$-modules, and using that

$$M \otimes_A B = M \otimes_{A/f} A/f \otimes_A B = M \otimes_{A/f} B/fB$$

for $f$-torsion $A$-modules $M$. The rest follows by devissage and the fact that $M \mapsto M \otimes_A B$ commutes with filtered colimits and is exact. \qed

Next, we prove a series of lemmas which tell us that the category $\text{Mod}(A)[f^\infty]$ is insensitive to passing to an $f$-adic neighbourhood. First, we need a nice presentation.

**Lemma 1.3.** Any module $M \in \text{Mod}(A)[f^\infty]$ admits a resolution $K. \to M$ with each $K_i$ a direct sum of copies of $A/f^n$ for $n$ variable.

**Proof.** For any $M \in \text{Mod}(A)[f^\infty]$, there is a canonical surjection

$$\bigoplus_{m \in M} A/f^n \to M \to 0$$

where $n_m$ is the smallest positive integer such that $f^n m \cdot m = 0$. The kernel of the preceding surjection is also an $f^\infty$-torsion module. Proceeding inductively, we construct a canonical resolution of $M$ by $A$-modules which are direct sums of copies of $A/f^n$ for variable $n$, as desired. \qed

Next, we show that passing to $f$-adic neighbourhoods does not change $f^\infty$-torsion modules.

**Lemma 1.4.** Let $\phi : A \to B$ be an $f$-adic neighbourhood. For any module $M \in \text{Mod}(A)[f^\infty]$, the natural map $M \mapsto M \otimes_A B$ is an isomorphism.

**Proof.** First assume that $M \in \text{Mod}(A)[f]$. In this case, $M$ is an $A/f$-module. Hence, we have an isomorphism

$$M \otimes_A B \simeq M \otimes_{A/f} B/fB \simeq M \otimes_{A/f} A/fA \simeq M$$

proving the claim. The general case follows by devissage. Indeed, using the isomorphism $A/fA \simeq B/fB$ and the flatness of $A \to B$, one shows that $A/f^n A \simeq B/f^n B$ for all $n \geq 0$. By the same argument as above, it follows that for any $A/f^n$-module $M$, the natural map $M \mapsto M \otimes_A B$ is bijective. Since any $M \in \text{Mod}(A)[f^\infty]$ can be written as a filtered colimit of $A/f^n$-modules for variable $n$, the claim follows from the fact that tensor products commute with colimits. \qed

We can now show that the category $\text{Mod}(A)[f^\infty]$ does not change on passing to an $f$-adic neighbourhood.

**Lemma 1.5.** Let $\phi : A \to B$ be an $f$-adic neighbourhood. Then the functor $M \mapsto M \otimes_A B$ defines an equivalence $\text{Mod}(A)[f^\infty] \to \text{Mod}(B)[f^\infty]$.

**Proof.** We first show full faithfulness. In fact, we will show that the natural map

$$\text{Hom}_A(M, N) \to \text{Hom}_B(M_B, N_B)$$
is an isomorphism if $M$ or $N$ is $f^\infty$-torsion. When $M$ is finitely presented, this follows from Lemma 1.4 once we observe that the formation of $\text{Hom}_A(M, N)$ commutes with flat base change on $A$. In general, we write $M$ as a filtered colimit $\colim_i M_i$ where each $M_i$ is finitely presented, and then use the following sequence of isomorphisms:

$$\text{Hom}_A(M, N) = \text{Hom}_A(\colim_i M_i, N) = \lim_i \text{Hom}_A(M_i, N) = \lim_i \text{Hom}_B(M_i, N_B) = \text{Hom}_B(\colim_i M_i, N_B) = \text{Hom}_B(M_B, N_B)$$

where the third equality uses the finitely presented case, while the last one uses the commutativity of $M \to M \otimes_A B$ with filtered colimits. In particular, the functor $\text{Mod}(A)[f^\infty] \to \text{Mod}(B)[f^\infty]$ is fully faithful.

For essential surjectivity, we simply note that for any $N \in \text{Mod}(B)[f^\infty]$, the natural map $N \otimes_A B \to N$ is an isomorphism by Lemma 1.4.

We can improve on the full faithfulness of Lemma 1.5 by showing that Ext-groups whose source lies in $\text{Mod}(A)[f^\infty]$ are insensitive to passing to $f$-adic neighbourhoods as well.

**Lemma 1.6.** Given $M \in \text{Mod}(A)[f^\infty]$ and $N \in \text{Mod}(A)$, the natural map

$$\text{Ext}^i_A(M, N) \to \text{Ext}^i_B(M_B, N_B)$$

is an isomorphism for all $i$.

**Proof.** We prove the statement by induction on $i$. The case $i = 0$ was proven in the course of Lemma 1.5. For larger $i$, using Lemma 1.3, one can immediately reduce to the case that $M = A/f^n$ for suitable $n$. In this case, we argue using a dimension shifting argument; the failure of $f$ to be regular element of $A$ forces us to introduce some derived notation. Let $K$ denote the two-term complex

$$A \xrightarrow{f^n} A.$$ 

In the derived category $D(\text{Mod}(A))$, there is an exact triangle of the form

$$K \to A/f^n[-1] \to A[f^n][1] \to K[1]$$

where $A[f^n]$ is kernel of multiplication by $f^n$ on $A$. Applying $\text{Ext}^i(-, N)$ then gives us a long exact sequence

$$\ldots \to \text{Ext}^{i-1}_A(A[f^n], N) \to \text{Ext}^{i+1}_A(A/f^n, N) \to \text{Ext}^i_A(K, N) \to \ldots$$

Induction on $i$ then reduces us to verifying that $\text{Ext}^i_A(K, N) \simeq \text{Ext}^i_B(K_B, N_B)$ for all $i$. The definition of $K$ gives us an exact triangle

$$A[-1] \to K \to A \xrightarrow{\delta} A$$

where the boundary map $\delta$ is identified with $f^n$, up to a sign. Using the projectivity of $A$, we see that $\text{Ext}^i_A(K, N) = 0$ for $i > 1$, and for $i \leq 1$ there is a short exact sequence

$$0 \to \text{Ext}^0_A(K, N) \to \text{Hom}(A, N) \xrightarrow{f^n} \text{Hom}(A, N) \to \text{Ext}^1_A(K, N) \to 0.$$ 

Hence, we may identify

$$\text{Ext}^0_A(K, N) = N[f^n] \quad \text{and} \quad \text{Ext}^1_A(K, N) = N/f^n N.$$ 

The formation of the groups $\text{Ext}^i_A(K, N)$ clearly commutes with base changing along $A \to B$. On the other hand, since $M \simeq M \otimes_A B$ for any $f^n$-torsion $A$-module (see Lemma 1.4), the right hand side of the preceding equalities does not change on base changing along $A \to B$. Thus, it follows that

$$\text{Ext}^i_A(K, N) \simeq \text{Ext}^i_B(K_B, N_B)$$

as desired. □

Lastly, we prove a couple of facts concerning the behaviour of $f$-torsionfree modules.

**Lemma 1.7.** Let $M$ be an $A$-module without $f$-torsion, and let $\phi : A \to B$ be an $f$-adically flat ring map. An element $m \in M$ is divisible by $f$ in the $A$-module $M$ if and only if the same is true for $m \otimes 1 \in M \otimes_A B$ in the $B$-module $M \otimes_A B$. 

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Proof. By hypothesis, there is a short exact sequence
\[ 0 \to M \xrightarrow{f} M \to M/fM \to 0. \]
Assume \( m \in M \) is not divisible by \( f \). Thus, the corresponding element in \( M/fM \) is not zero. By the faithful flatness of \( A/fA \to B/fB \), the resulting element of \( M/fM \otimes_A B \simeq M \otimes_A B/(M \otimes_A B) \) is also non-zero, which implies the result. \( \square \)

**Lemma 1.8.** Let \( M \) be an \( A \)-module without \( f \)-torsion. Then the natural map \( M \to M_f \) is injective.

Proof. The kernel of \( M \to M_f \) is spanned by elements \( m \in M \) satisfying \( f^n m = 0 \) for some \( n > 0 \). The hypothesis on \( M \) and an easy induction on \( n \) imply that \( m = 0 \). \( \square \)

2. THE FULL FAITHFULNESS

In this section, we establish the full faithfulness of the functor \( \mathcal{F} \) of Theorem 0.2. Like in the previous section, we fix the ring \( A \) and the element \( f \) under consideration. First, we show that an object in \( \text{Mod}(A)[f^\infty] \) is determined by the glueing data it determines.

**Lemma 2.1.** Let \( M \) be an \( f^\infty \)-torsion \( A \)-module. Let \( \phi : A \to B \) be an \( f \)-adic neighbourhood. Then the natural map
\[ M \to M_f \times_{M_{B_f}} M_B \]
is an isomorphism.

Proof. The hypothesis implies that \( M_f = M_{B_f} = 0 \). It then suffices to check that \( M \simeq M_B \), which follows from Lemma 1.4. \( \square \)

Next, we show that an \( f \)-torsionfree module is determined by the glueing data it determines.

**Lemma 2.2.** Let \( M \) be an \( A \)-module without \( f \)-torsion, and let \( \phi : A \to B \) be an \( f \)-adically faithfully flat ring map. Then the natural map
\[ M \to M_f \times_{M_{B_f}} M_B \]
is an isomorphism.

Proof. As \( M \) has no \( f \)-torsion, the same is true for \( M \otimes_A B \). Thus, the vertical maps in the diagram
\[
\begin{array}{c}
M \\
\downarrow \\
M_f \\
\downarrow \\
M_{B_f}
\end{array}
\to
\begin{array}{c}
M_B \\
\downarrow \\
M_{B_f}
\end{array}
\]
are injective. We may therefore view \( M \) as being an \( A \)-submodule of \( M_f \), and similarly for \( M_B \). It follows then that the map \( M \to M_f \times_{M_{B_f}} M_B \) is injective. For surjectivity, let \((x, y) \in M_f \times_{M_{B_f}} M_B \) be an element. Then there exists an \( n \) such that \( f^n x = m \in M \). The image of \( m \) in \( M_B \) agrees with \( f^n y \) as both these elements have the same image in \( M_{B_f} \). Thus, the element \( m \) is divisible by \( f^n \) in \( M_B \). By Lemma 1.7, the element \( m \in M \) is divisible by \( f^n \) in \( M \) itself. Thus, we may write \( m = f^n x' \). Since \( f \) acts invertibly on \( M_f \), it follows that \( x = x' \in M_f \), and thus the element \( x \in M_f \) actually comes from \( M \). One can then easily check that the image of \( x = x' \) in \( M_B \) agrees with \( y \) (as the same is true in \( M_{B_f} \)). Thus, the element \( x' \in M \) maps to \((x, y)\) as desired. \( \square \)

Combining the previous two cases, we verify that arbitrary modules are determined by their glueing data.

**Lemma 2.3.** Let \( M \) be an \( A \)-module, and let \( \phi : A \to B \) be an \( f \)-adic neighbourhood. Then the natural map
\[ M \to M_f \times_{M_{B_f}} M_B \]
is an isomorphism.

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Proof. Given an $A$-module $M$, let $T \subset M$ denote its $f^\infty$-torsion. Then we have an exact sequence

$$0 \to T \to M \to N \to 0$$

with $N = M/T$ without $f$-torsion. It is also easy to see that the functor $M \mapsto M_f \times_{MB_f} MB$ is left exact, i.e., preserves finite limits. Thus, we may apply it to the preceding short exact sequence to obtain a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & T & \to & M & \to & N & \to & 0 \\
\downarrow{a} & & \downarrow{b} & & \downarrow{c} & & \\
0 & \to & T_f \times_{TB_f} TB & \to & M_f \times_{MB_f} MB & \to & N_f \times_{NB_f} NB & \to & 0
\end{array}
$$

with exact rows. The map $a$ is an isomorphism by Lemma 2.1, while the map $c$ is an isomorphism by Lemma 2.2. An easy diagram chase then shows that $b$ is also an isomorphism, as desired. \qed

Using the preceding results, we can prove the full faithfulness of $\mathcal{F}$.

Lemma 2.4. Let $M$ and $N$ be two $A$-modules, and let $\phi : A \to B$ be an $f$-adic neighbourhood. The natural map

$$a : \text{Hom}_A(M, N) \cong \text{Hom}_{A_f}(M_f, N_f) \times_{\text{Hom}_{B_f}(MB_f, NB_f)} \text{Hom}_B(MB, NB)$$

is an isomorphism. Thus, the functor $\mathcal{F}$ is fully faithful.

Proof. The injectivity of $a$ immediately follows from Lemma 2.3. Conversely, given maps $g_f : M_f \to N_f$ and $g_B : MB \to NB$ defining the same map over $B_f$, we obtain an induced map $g : M \to N$ via Lemma 2.3. Subtracting the map $g$ induces from $g_f$ and $g_B$, we may assume that both $g_f$ and $g_B$ induce the $0$ map $M \to N$. It suffices to show that in this case $g_f$ and $g_B$ are both $0$. However, this is clear since both $M_f$ and $MB$ are generated by $M$.

3. Essential surjectivity

We first recall the general definition of a fibre product of categories.

Definition 3.1. Given a diagram

$$
\begin{array}{ccc}
\mathcal{A} & \to & \mathcal{C} \\
\downarrow{\mathcal{B}} & & \\
\mathcal{C}
\end{array}
$$

of categories, we define the fibre product $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ to be the category of triples $(a, b, f)$ where $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $f$ is an isomorphism in $\mathcal{C}$ between the images of $a$ and $b$; morphisms are defined in the obvious way.

Remark 3.2. In the situation considered above, the fibre product $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ inherits properties as well as structures present on $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ that are preserved by the functors. For example, if all three categories are abelian $\otimes$-categories with the functors being exact and $\otimes$-preserving, then the fibre product $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ also inherits the structure of an abelian $\otimes$-category; this will be the case in the example we consider.

We place ourselves back in the situation of Theorem 0.2, i.e., we fix a ring $A$, an element $f \in A$, and an $f$-adic neighbourhood $\phi : A \to B$. Since both $B$ and $A_f$ are flat $A$-algebras, the fibre product $\text{Mod}(A_f) \times_{\text{Mod}(B_f)} \text{Mod}(B)$ is an abelian category with a natural $\otimes$-structure. Moreover, base changing defines the functor

$$\mathcal{F} : \text{Mod}(A) \to \text{Mod}(A_f) \times_{\text{Mod}(B_f)} \text{Mod}(B)$$

which is easily checked to preserve the $\otimes$-structure. We will show that $\mathcal{F}$ is an equivalence; the full faithfulness was established in Lemma 2.4. First, we show that $\mathcal{F}$ has nice colimit properties.

Lemma 3.3. The functor $\mathcal{F}$ is exact and commutes with arbitrary colimits.

Proof. The exactness follows from the $A$-flatness of $A_f$ and $B$, while the cocontinuity is a general fact about tensor products. \qed

Next, we verify that objects in the category of glueing data admit a nice presentation in terms of actual $A$-modules.
Lemma 3.4. Given an object \((M_1, M_2, \psi) \in \text{Mod}(A_f) \times_{\text{Mod}(B_f)} \text{Mod}(B)\), there exists an \(A\)-module \(P\), an \(f^\infty\)-torsion \(A\)-module \(Q\), and a right exact sequence

\[ \mathcal{F}(P) \to (M_1, M_2, \psi) \to \mathcal{F}(Q) \to 0 \]

in the category \(\text{Mod}(A_f) \times_{\text{Mod}(B_f)} \text{Mod}(B)\).

Proof. Let \((M_1, M_2, \psi)\) be as above. For an \(x \in M_1\), let \(n_x\) be the minimal positive integer such that the image of \(f^{n_x} \cdot x\) in \(M_1 \otimes_{A_f} B_f \simeq M_2 \otimes_{B_f} B_f\) lifts to an element \(y_x\) in \(M_2\). The choice of such a lift \(y_x\) defines a morphism \(\mathcal{F}(A) \to (M_1, M_2, \psi)\) via \(f^{n_x} x\) on the first factor, and \(y_x\) on the second factor. Thus, after fixing a lift \(y_x\) of \(f^{n_x} x\) for each \(x \in M_1\), we obtain a morphism

\[ \oplus_{x \in M_1} \mathcal{F}(A) \xrightarrow{T} (M_1, M_2, \psi). \]

The first component of this map is surjective because \(f\) is a unit in \(M_1\). Thus, the cokernel is of the form \((0, Q, 0)\) for some \(Q \in \text{Mod}(B)\). Moreover, since \(Q \otimes_{B_f} B_f = 0\), we have \(Q \in \text{Mod}(B)[f^\infty]\). By Lemma 1.5, it follows that \((0, Q, 0) \simeq \mathcal{F}(Q)\) where second term is defined by viewing \(Q\) as an \(A\)-module in the obvious way. Thus, we obtain an exact sequence

\[ \oplus_{x \in M_1} \mathcal{F}(A) \xrightarrow{T} (M_1, M_2, \psi) \to \mathcal{F}(Q) \to 0. \]

Since the functor \(\mathcal{F}\) commutes with colimits (see Lemma 3.3), we can absorb the coproduct on the left to rewrite the above sequence as

\[ \mathcal{F}(P) \to (M_1, M_2, \psi) \to \mathcal{F}(Q) \to 0 \]

with \(P \in \text{Mod}(A)\), and \(Q \in \text{Mod}(A)[f^\infty]\) as desired. \(\square\)

We need the following abstract fact about abelian categories to finish the proof.

Lemma 3.5. Let \(F : A \to B\) be an exact fully faithful functor between abelian categories \(A\) and \(B\), and let \(A' \subset A\) be a full abelian subcategory of \(A\). Assume that \(F\) induces an isomorphism \(\text{Ext}_A^1(a_1, a_2) \to \text{Ext}_B^1(F(a_1), F(a_2))\) when \(a_1 \in A'\) and \(a_2 \in A\) (where the \(\text{Ext}\) groups being considered are the Yoneda ones). Further, assume that for every object \(b \in B\), there exist objects \(a \in A\), and \(a' \in A' \subset A\), and a right exact sequence

\[ F(a) \to b \to F(a') \to 0. \]

Then \(F\) is an equivalence.

Proof. It suffices to show that \(F\) is essentially surjective. Given \(b_0 \in B\), choose \(a_0 \in A\) and \(a'_0 \in A'\) and an exact sequence

\[ 0 \to b_1 \to F(a_0) \to b_0 \to F(a'_0) \to 0 \]

where \(b_1 \in B\) is the kernel of \(F(a_0) \to b_0\). Applying the same procedure to \(b_1\), we can find \(a_1 \in A\), \(a'_1 \in A'\), and an exact sequence

\[ F(a_1) \to b_1 \to F(a'_1) \to 0. \]

Since the map \(b_1 \to F(a_0) \to b_0\) is 0, the same is true for the map \(F(a_1) \to b_1 \to F(a_0) \to b_0\). Thus, we obtain a sequence

\[ F(a'_1) = b_1/F(a_1) \to F(a_0)/\text{im}(F(a_1)) \to b_0 \to F(a'_0) \to 0. \]

The object \(F(a_0)/\text{im}(F(a_1))\) is isomorphic to an object of the form \(F(a_2)\) for some \(a_2 \in A\) as the functor \(F\) is fully faithful and exact. Thus, we may rewrite the above sequence as

\[ F(a'_1) \to F(a_2) \to b_0 \to F(a'_0) \to 0. \]

The same reasoning as above shows that \(F(a_2)/\text{im}(F(a'_1))\) is isomorphic to an object of the form \(F(a_3)\) for some \(a_3 \in A\). Thus, we obtain a short exact sequence

\[ 0 \to F(a_3) \to b_0 \to F(a'_0) \to 0 \]

which realises \(b_0\) as an extension of \(F(a'_0)\) by \(F(a_3)\). Since \(a'_0 \in A'\), we know that all such extensions lie in the essential image of \(F\) by assumption. Thus, so does \(b_0\), as desired. \(\square\)

We now observe that the proof is complete.

Proof of Theorem 0.2. Theorem 0.2 follows formally from Lemma 3.5, Lemma 3.4, Lemma 2.4, and Lemma 1.6. \(\square\)