FORMAL GLUEING OF MODULE CATEGORIES

BHARGAV BHATT

Fix a noetherian scheme X, and a closed subscheme Z with complement U. Our goal is to explain a result of Artin that describes how coherent sheaves on X can be constructed (uniquely) from coherent sheaves on the formal completion of X along Z, and those on U with a suitable compatibility on the overlap. In fact, the main result is a general (i.e., non-noetherian) local version, which we will state once we have the following definition in place.

Definition 0.1. Given a ring A and an element $f \in A$, a ring map $\phi : A \to B$ is said to be *f-adically faithfully flat* if ϕ is flat, and $\phi/f : A/fA \to B/fB$ is faithfully flat. The map $\phi : A \to B$ is said to be *an f-adic neighbourhood* if it is f-adically faithfully flat, and the induced map $A/fA \to B/fB$ is an isomorphism.

We let $\mathrm{Mod}(A)$ denote the abelian category of A-modules over a ring A, while $\mathrm{Mod}_{\mathrm{fg}}(A)$ denotes the subcategory of finitely generated A-modules. The main result is:

Theorem 0.2. *Let A be a ring, and let $f \in A$. Let $\phi : A \to B$ be an f-adic neighbourhood. Then the natural map*

$$\mathcal{F} : \mathrm{Mod}(A) \to \mathrm{Mod}(A_f) \times_{\mathrm{Mod}(B_f)} \mathrm{Mod}(B)$$

is an equivalence.

The category $\mathrm{Mod}(A_f) \times_{\mathrm{Mod}(B_f)} \mathrm{Mod}(B)$ appearing on the right side of the expression in Theorem 0.2 is the category of triples (M_1, M_2, ψ) where M_1 is an A_f-module, M_2 is a B-module, and $\psi : M_1 \otimes_{A_f} B_f \simeq M_2 \otimes_B B_f$ is a B_f-isomorphism. The natural map referred to in Theorem 0.2 is defined by $\mathcal{F}(M) = (M_f, M_B, \mathrm{can})$ where $\mathrm{can} : M_f \otimes_{A_f} B_f \simeq M_B \otimes_B B_f$ is the natural isomorphism. We generally refer to objects of this category as "glueing data." The motivation behind this terminology is topological and explained in Remark 0.5.

A useful special case of Theorem 0.2 is when A is noetherian, and B is a completion of A at an element f. The completion $A \to B$ is flat by basic theorems in noetherian ring theory, and the functor $M \mapsto M \otimes_A B$ can be identified with the f-adic completion functor when M is finitely generated. Thus, we obtain:

Corollary 0.3. *Let A be a noetherian ring, let $f \in A$ be an element, and let $\hat{A}$ be the f-adic completion of A. Then the obvious functors (localisation and completion) define an equivalence*

$$\mathrm{Mod}_{\mathrm{fg}}(A) \simeq \mathrm{Mod}_{\mathrm{fg}}(A_f) \times_{\mathrm{Mod}_{\mathrm{fg}}(\hat{A}_f)} \mathrm{Mod}_{\mathrm{fg}}(\hat{A})$$

Remark 0.4. The equivalence of Theorem 0.2 preserves the obvious $\otimes$-structure on either side. Thus, it defines equivalences of various categories built out of the pair $(\mathrm{Mod}(A), \otimes)$, such as the category of A-algebras.

Remark 0.5. Theorem 0.2 may be regarded as an algebraic analogue of the following trivial theorem from topology: given a manifold X with a closed submanifold Z having complement U, specifying a sheaf on X is the same as specifying a sheaf on U, a sheaf on an unspecified tubular neighbourhood T of Z in X, and an isomorphism between the two resulting sheaves along $T \cap U$. The lack of tubular neighbourhoods in algebraic geometry forces us to work with formal neighbourhoods instead, rendering the proof a little more complicated.

Remark 0.6. We suspect that Theorem 0.2 follows formally from the existence of a good model structure for the flat topology. Specifically, if one has a model structure where open immersions are cofibrations, then the square

$$\begin{tikzcd} \mathrm{Spec}(B_f) \arrow[r] \arrow[d] & \mathrm{Spec}(B) \arrow[d] \\ \mathrm{Spec}(A_f) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

will be a homotopy pushout square. Evaluating the fpqc-stack $\mathrm{Mod}(-)$ on this pushout diagram would then allow us to deduce Theorem 0.2 from usual fpqc-descent.

1. Generalities

Fix a ring A and an element f.

Definition 1.1. An A-module M said to be *an f^∞-torsion A-module* if for each $m \in M$, there exists an $n > 0$ such that $f^n m = 0$. The full subcategory of $\mathrm{Mod}(A)$ spanned by f^∞-torsion modules is denoted $\mathrm{Mod}(A)[f^\infty]$, while the subcategory spanned by f^n-torsion modules is denoted $\mathrm{Mod}(A)[f^n]$.

We first reformulate the definition of f-adic faithful flatness in terms of the category $\mathrm{Mod}(A)[f^\infty]$.

Lemma 1.2. *Fix a ring map $\phi : A \to B$. Then the following are equivalent*

1. *The map ϕ is f-adically faithfully flat.*
2. *The map ϕ is flat, and the map $\mathrm{Spec}(B/fB) \to \mathrm{Spec}(A/fA)$ is surjective.*
3. *The map ϕ is flat, and the functor $M \mapsto M \otimes_A B$ is faithful on $\mathrm{Mod}(A)[f]$.*
4. *The map ϕ is flat, and the functor $M \mapsto M \otimes_A B$ is faithful on $\mathrm{Mod}(A)[f^\infty]$.*

Proof. (1) and (2) being equivalent is standard, while the equivalence of either with (3) follows by identifying f-torsion A-modules with A/f-modules, and using that

$$M \otimes_A B = M \otimes_{A/f} A/f \otimes_A B = M \otimes_{A/f} B/fB$$

for f-torsion A-modules M. The rest follows by devissage and the fact that $M \mapsto M \otimes_A B$ commutes with filtered colimits and is exact. □

Next, we prove a series of lemmas which tell us that the category $\mathrm{Mod}(A)[f^\infty]$ is insensitive to passing to an f-adic neighbourhood. First, we need a nice presentation.

Lemma 1.3. *Any module $M \in \mathrm{Mod}(A)[f^\infty]$ admits a resolution $K. \to M$ with each K_i a direct sum of copies of A/f^n for n variable.*

Proof. For any $M \in \mathrm{Mod}(A)[f^\infty]$, there is a canonical surjection

$$\oplus_{m \in M} A/f^{n_m} \to M \to 0$$

where n_m is the smallest positive integer such that $f^{n_m} \cdot m = 0$. The kernel of the preceding surjection is also an f^∞-torsion module. Proceeding inductively, we construct a canonical resolution of M by A-modules which are direct sums of copies of A/f^n for variable n, as desired. □

Next, we show that passing to f-adic neighbourhoods does not change f^∞-torsion modules.

Lemma 1.4. *Let $\phi : A \to B$ be an f-adic neighbourhood. For any module $M \in \mathrm{Mod}(A)[f^\infty]$, the natural map $M \to M \otimes_A B$ is an isomorphism.*

Proof. First assume that $M \in \mathrm{Mod}(A)[f]$. In this case, M is an A/f-module. Hence, we have an isomorphism

$$M \otimes_A B \simeq M \otimes_{A/f} B/fB \simeq M \otimes_{A/fA} A/fA \simeq M$$

proving the claim. The general case follows by devissage. Indeed, using the isomorphism $A/fA \simeq B/fB$ and the flatness of $A \to B$, one shows that $A/f^nA \simeq B/f^nB$ for all $n \geq 0$. By the same argument as above, it follows that for any A/f^n-module M, the natural map $M \to M \otimes_A B$ is bijective. Since any $M \in \mathrm{Mod}(A)[f^\infty]$ can be written as a filtered colimit of A/f^n-modules for variable n, the claim follows from the fact that tensor products commute with colimits. □

We can now show that the category $\mathrm{Mod}(A)[f^\infty]$ does not change on passing to an f-adic neighbourhood.

Lemma 1.5. *Let $\phi : A \to B$ be an f-adic neighbourhood. Then the functor $M \mapsto M \otimes_A B$ defines an equivalence* $\mathrm{Mod}(A)[f^\infty] \to \mathrm{Mod}(B)[f^\infty]$.

Proof. We first show full faithfulness. In fact, we will show that the natural map

$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_B(M_B, N_B)$$

is an isomorphism if M or N is f^∞-torsion. When M is finitely presented, this follows from Lemma 1.4 once we observe that the formation of $\mathrm{Hom}_A(M,N)$ commutes with flat base change on A. In general, we write M as a filtered colimit $\mathrm{colim}_i\, M_i$ where each M_i is finitely presented, and then use the following sequence of isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_A(M,N) &= \mathrm{Hom}_A(\mathrm{colim}_i\, M_i, N) \\
&= \lim_i \mathrm{Hom}_A(M_i, N) \\
&= \lim_i \mathrm{Hom}_B(M_{i,B}, N_B) \\
&= \mathrm{Hom}_B(\mathrm{colim}_i\, M_{i,B}, N_B) \\
&= \mathrm{Hom}_B(M_B, N_B)
\end{aligned}$$

where the third equality uses the finitely presented case, while the last one uses the commutativity of $M \mapsto M \otimes_A B$ with filtered colimits. In particular, the functor $\mathrm{Mod}(A)[f^\infty] \to \mathrm{Mod}(B)[f^\infty]$ is fully faithful.

For essential surjectivity, we simply note that for any $N \in \mathrm{Mod}(B)[f^\infty]$, the natural map $N \otimes_A B \to N$ is an isomorphism by Lemma 1.4, □

We can improve on the full faithfulness of Lemma 1.5 by showing that Ext-groups whose source lies in $\mathrm{Mod}(A)[f^\infty]$ are insensitive to passing to f-adic neighbourhoods as well.

Lemma 1.6. *Given $M \in \mathrm{Mod}(A)[f^\infty]$ and $N \in \mathrm{Mod}(A)$, the natural map*

$$\mathrm{Ext}^i_A(M,N) \to \mathrm{Ext}^i_B(M_B, N_B)$$

is an isomorphism for all i.

Proof. We prove the statement by induction on i. The case $i=0$ was proven in the course of Lemma 1.5. For larger i, using Lemma 1.3, one can immediately reduce to the case that $M = A/f^n$ for suitable n. In this case, we argue using a dimension shifting argument; the failure of f to be regular element of A forces us to introduce some derived notation. Let K denote the two-term complex

$$A \xrightarrow{f^n} A.$$

In the derived category $D(\mathrm{Mod}(A))$, there is an exact triangle of the form

$$K \to A/f^n[-1] \to A[f^n][1] \to K[1]$$

where $A[f^n]$ is kernel of multiplication by f^n on A. Applying $\mathrm{Ext}^i(-,N)$ then gives us a long exact sequence

$$\dots \mathrm{Ext}^{i-1}_A(A[f^n],N) \to \mathrm{Ext}^{i+1}_A(A/f^n,N) \to \mathrm{Ext}^i_A(K,N)\dots$$

Induction on i then reduces us to verifying that $\mathrm{Ext}^i_A(K,N) \simeq \mathrm{Ext}^i_B(K_B,N_B)$ for all i. The definition of K gives us an exact triangle

$$A[-1] \to K \to A \xrightarrow{\delta} A$$

where the boundary map δ is identified with f^n, up to a sign. Using the projectivity of A, we see that $\mathrm{Ext}^i_A(K,N) = 0$ for $i > 1$, and for $i \leq 1$ there is a short exact sequence

$$0 \to \mathrm{Ext}^0_A(K,N) \to \mathrm{Hom}(A,N) \xrightarrow{f^n} \mathrm{Hom}(A,N) \to \mathrm{Ext}^1_A(K,N) \to 0.$$

Hence, we may identify

$$\mathrm{Ext}^0_A(K,N) = N[f^n] \quad \text{and} \quad \mathrm{Ext}^1_A(K,N) = N/f^nN.$$

The formation of the groups $\mathrm{Ext}^i_A(K,N)$ clearly commutes with base changing along $A \to B$. On the other hand, since $M \simeq M \otimes_A B$ for any f^n-torsion A-module (see Lemma 1.4), the right hand side of the preceding equalities does not change on base changing along $A \to B$. Thus, it follows that

$$\mathrm{Ext}^i_A(K,N) \simeq \mathrm{Ext}^i_B(K_B,N_B)$$

as desired. □

Lastly, we prove a couple of facts concerning the behaviour of f-torsionfree modules.

Lemma 1.7. *Let M be an A-module without f-torsion, and let $\phi: A \to B$ be an f-adically flat ring map. An element $m \in M$ is divisible by f in the A-module M if and only if the same is true for $m \otimes 1 \in M \otimes_A B$ in the B-module $M \otimes_A B$.*

Proof. By hypothesis, there is a short exact sequence

$$0 \to M \xrightarrow{f} M \to M/fM \to 0.$$

Assume $m \in M$ is not divisible by f. Thus, the corresponding element in M/fM is not zero. By the faithful flatness of $A/fA \to B/fB$, the resulting elemnt of $M/fM \otimes_A B \simeq M \otimes_A B/f(M \otimes_A B)$ is also non-zero, which implies the result. □

Lemma 1.8. *Let M be an A-module without f-torsion. Then the natural map $M \to M_f$ is injective.*

Proof. The kernel of $M \to M_f$ is spanned by elements $m \in M$ satisfying $f^n m = 0$ for some $n > 0$. The hypothesis on M and an easy induction on n imply that $m = 0$. □

2. The full faithfulness

In this section, we establish the full faithfulness of the functor $\mathcal{F}$ of Theorem 0.2. Like in the previous section, we fix the ring A and the element f under consideration. First, we show that an object in $\mathrm{Mod}(A)[f^\infty]$ is determined by the glueing data it determines.

Lemma 2.1. *Let M be an f^∞-torsion A-module. Let $\phi : A \to B$ be an f-adic neighbourhood. Then the natural map*

$$M \to M_f \times_{M_{B_f}} M_B$$

is an isomorphism.

Proof. The hypothesis implies that $M_f = M_{B_f} = 0$. It then suffices to check that $M \simeq M_B$, which follows from Lemma 1.4. □

Next, we show that an f-torsionfree module is determined by the glueing data it determines.

Lemma 2.2. *Let M be an A-module without f-torsion, and let $\phi : A \to B$ be an f-adically faithfully flat ring map. Then the natural map*

$$M \to M_f \times_{M_{B_f}} M_B$$

is an isomorphism.

Proof. As M has no f-torsion, the same is true for $M \otimes_A B$. Thus, the vertical maps in the diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M_B \arrow[d] \\ M_f \arrow[r] & M_{B_f} \end{tikzcd}$$

are injective. We may therefore view M as being an A-submodule of M_f, and similarly for M_B. It follows then that the map $M \to M_f \times_{M_{B_f}} M_B$ is injective. For surjectivity, let $(x, y) \in M_f \times_{M_{B_f}} M_B$ be an element. Then there exists an n such that $f^n x = m \in M$. The image of m in M_B agrees with $f^n y$ as both these elements have the same image in M_{B_f}. Thus, the element m is divisible by f^n in M_B. By Lemma 1.7, the element $m \in M$ is divisible by f^n in M itself. Thus, we may write $m = f^n x'$. Since f acts invertibly on M_f, it follows that $x = x' \in M_f$, and thus the element $x \in M_f$ actually comes from M. One can then easily check that the image of $x = x'$ in M_B agrees with y (as the same is true in M_{B_f}). Thus, the element $x' \in M$ maps to (x, y) as desired. □

Combining the previous two cases, we verify that arbitrary modules are determined by their glueing data.

Lemma 2.3. *Let M be an A-module, and let $\phi : A \to B$ be an f-adic neighbourhood. Then the natural map*

$$M \to M_f \times_{M_{B_f}} M_B$$

is an isomorphism.

Proof. Given an A-module M, let $T \subset M$ denote its f^∞-torsion. Then we have an exact sequence

$$0 \to T \to M \to N \to 0$$

with $N = M/T$ without f-torsion. It is also easy to see that the functor $M \mapsto M_f \times_{M_{B_f}} M_B$ is left exact, i.e., preserves finite limits. Thus, we may apply it to the preceding short exact sequence to obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r] \arrow[d, "a"] & M \arrow[r] \arrow[d, "b"] & N \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & T_f \times_{T_{B_f}} T_B \arrow[r] & M_f \times_{M_{B_f}} M_B \arrow[r] & N_f \times_{N_{B_f}} N_B & \end{tikzcd}$$

with exact rows. The map a is an isomorphism by Lemma 2.1, while the map c is an isomorphism by Lemma 2.2. An easy diagram chase then shows that b is also an isomorphism, as desired. □

Using the preceding results, we can prove the full faithfulness of $\mathcal{F}$.

Lemma 2.4. *Let M and N be two A-modules, and let $\phi : A \to B$ be an f-adic neighbourhood. The natural map*

$$a : \mathrm{Hom}_A(M, N) \simeq \mathrm{Hom}_{A_f}(M_f, N_f) \times_{\mathrm{Hom}_{B_f}(M_{B_f}, N_{B_f})} \mathrm{Hom}_B(M_B, N_B)$$

is an isomorphism. Thus, the functor $\mathcal{F}$ is fully faithful.

Proof. The injectivity of a immediately follows from Lemma 2.3. Conversely, given maps $g_f : M_f \to N_f$ and $g_B : M_B \to N_B$ defining the same map over B_f, we obtain an induced map $g : M \to N$ via Lemma 2.3. Subtracting the map g induces from g_f and g_B, we may assume that both g_f and g_B induce the 0 map $M \to N$. It suffices to show that in this case g_f and g_B are both 0. However, this is clear since both M_f and M_B are generated by M.

□

3. Essential surjectivity

We first recall the general definition of a fibre product of categories.

Definition 3.1. Given a diagram

$$\begin{tikzcd} & \mathcal{A} \arrow[d] \\ \mathcal{B} \arrow[r] & \mathcal{C} \end{tikzcd}$$

of categories, we define the fibre product $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ to be the category of tripes (a, b, f) where $a \in \mathcal{A}$, $b \in \mathcal{B}$, and f is an isomorphism in $\mathcal{C}$ between the images of a and b; morphisms are defined in the obvious way.

Remark 3.2. In the situation considered above, the fibre product $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ inherits properties as well as structures present on $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ that are preserved by the functors. For example, if all three categories are abelian $\otimes$-categories with the functors being exact and $\otimes$-preserving, then the fibre product $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ also inherits the structure of an abelian $\otimes$-category; this will be the case in the example we consider.

We place ourselves back in the situation of Theorem 0.2, i.e., we fix a ring A, an element $f \in A$, and an f-adic neighbourhood $\phi : A \to B$. Since both B and A_f are flat A-algebras, the fibre product $\mathrm{Mod}(A_f) \times_{\mathrm{Mod}(B_f)} \mathrm{Mod}(B)$ is an abelian category with a natural $\otimes$-structure. Moreover, base changing defines the functor

$$\mathcal{F} : \mathrm{Mod}(A) \to \mathrm{Mod}(A_f) \times_{\mathrm{Mod}(B_f)} \mathrm{Mod}(B)$$

which is easily checked to preserve the $\otimes$-structure. We will show that $\mathcal{F}$ is an equivalence; the full faithfulness was established in Lemma 2.4. First, we show that $\mathcal{F}$ has nice cocontinuity properties.

Lemma 3.3. *The functor $\mathcal{F}$ is exact and commutes with arbitrary colimits.*

Proof. The exactness follows from the A-flatness of A_f and B, while the cocontinuity is a general fact about tensor products. □

Next, we verify that objects in the category of glueing data admit a nice presentation in terms of actual A-modules.

Lemma 3.4. *Given an object $(M_1, M_2, \psi) \in \mathrm{Mod}(A_f) \times_{\mathrm{Mod}(B_f)} \mathrm{Mod}(B)$, there exists a A-module P, an f^∞-torsion A-module Q, and a right exact sequence*

$$\mathcal{F}(P) \to (M_1, M_2, \psi) \to \mathcal{F}(Q) \to 0$$

in the category $\mathrm{Mod}(A_f) \times_{\mathrm{Mod}(B_f)} \mathrm{Mod}(B)$.

Proof. Let (M_1, M_2, ψ) be as above. For an $x \in M_1$, let n_x be the minimal positive integer such that the image of $f^{n_x} \cdot x$ in $M_1 \otimes_{A_f} B_f \simeq M_2 \otimes_B B_f$ lifts to an element y_x in M_2. The choice of such a lift y_x defines a morphism $\mathcal{F}(A) \to (M_1, M_2, \psi)$ via $f^{n_x} x$ on the first factor, and y_x on the second factor. Thus, after fixing a lift y_x of $f^{n_x} x$ for each $x \in M_1$, we obtain a morphism

$$\oplus_{x \in M_1} \mathcal{F}(A) \xrightarrow{T} (M_1, M_2, \psi).$$

The first component of this map is surjective because f is a unit in M_1. Thus, the cokernel is of the form $(0, Q, 0)$ for some $Q \in \mathrm{Mod}(B)$. Moreover, since $Q \otimes_B B_f = 0$, we have $Q \in \mathrm{Mod}(B)[f^\infty]$. By Lemma 1.5, it follows that $(0, Q, 0) \simeq \mathcal{F}(Q)$ where second term is defined by viewing Q as an A-module in the obvious way. Thus, we obtain an exact sequence

$$\oplus_{x \in M_1} \mathcal{F}(A) \xrightarrow{T} (M_1, M_2, \psi) \to \mathcal{F}(Q) \to 0.$$

Since the functor $\mathcal{F}$ commutes with colimits (see Lemma 3.3), we can absorb the coproduct on the left to rewrite the above sequence as

$$\mathcal{F}(P) \to (M_1, M_2, \psi) \to \mathcal{F}(Q) \to 0$$

with $P \in \mathrm{Mod}(A)$, and $Q \in \mathrm{Mod}(A)[f^\infty]$ as desired. □

We need the following abstract fact about abelian categories to finish the proof.

Lemma 3.5. *Let $F : \mathcal{A} \to \mathcal{B}$ be an exact fully faithful functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$, and let $\mathcal{A}' \subset \mathcal{A}$ be a full abelian subcategory of $\mathcal{A}$. Assume that F induces an isomorphism $\mathrm{Ext}^1_{\mathcal{A}}(a_1, a_2) \to \mathrm{Ext}^1_{\mathcal{B}}(F(a_1), F(a_2))$ when $a_1 \in \mathcal{A}'$ and $a_2 \in \mathcal{A}$ (where the Ext groups being considered are the Yoneda ones). Further, assume that for every object $b \in \mathcal{B}$, there exist objects $a \in \mathcal{A}$, and $a' \in \mathcal{A}' \subset \mathcal{A}$, and a right exact sequence*

$$F(a) \to b \to F(a') \to 0.$$

Then F is an equivalence.

Proof. It suffices to show that F is essentially surjective. Given $b_0 \in \mathcal{B}$, choose $a_0 \in \mathcal{A}$ and $a'_0 \in \mathcal{A}'$ and an exact sequence

$$0 \to b_1 \to F(a_0) \to b_0 \to F(a'_0) \to 0$$

where $b_1 \in \mathcal{B}$ is the kernel of $F(a_0) \to b_0$. Applying the same procedure to b_1, we can find $a_1 \in \mathcal{A}$, $a'_1 \in \mathcal{A}'$, and an exact sequence

$$F(a_1) \to b_1 \to F(a'_1) \to 0.$$

Since the map $b_1 \to F(a_0) \to b_0$ is 0, the same is true for the map $F(a_1) \to b_1 \to F(a_0) \to b_0$. Thus, we obtain a sequence

$$F(a'_1) = b_1/F(a_1) \to F(a_0)/\mathrm{im}(F(a_1)) \to b_0 \to F(a'_0) \to 0.$$

The object $F(a_0)/\mathrm{im}(F(a_1))$ is isomorphic to an object of the form $F(a_2)$ for some $a_2 \in \mathcal{A}$ as the functor F is fully faithful and exact. Thus, we may rewrite the above sequence as

$$F(a'_1) \to F(a_2) \to b_0 \to F(a'_0) \to 0.$$

The same reasoning as above shows that $F(a_2)/\mathrm{im}(F(a'_1))$ is isomorphic to an object of the form $F(a_3)$ for some $a_3 \in \mathcal{A}$. Thus, we obtain a short exact sequence

$$0 \to F(a_3) \to b_0 \to F(a'_0) \to 0$$

which realises b_0 as an extension of $F(a'_0)$ by $F(a_3)$. Since $a'_0 \in \mathcal{A}'$, we know that all such extensions lie in the essential image of F by assumption. Thus, so does b_0, as desired. □

We now observe that the proof is complete.

Proof of Theorem 0.2. Theorem 0.2 follows formally from Lemma 3.5, Lemma 3.4, Lemma 2.4, and Lemma 1.6. □