A FLAT MAP THAT IS NOT A DIRECTED LIMIT OF FINITELY PRESENTED FLAT MAPS

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The goal of this note is to show:

**Proposition 0.1.** There exists a commutative ring $A$ and a flat $A$-algebra $B$ which cannot be written as a filtered colimit of finitely presented flat $A$-algebras. In fact, we may choose $A$ to be a finite type $\mathbb{Z}$-algebra.

For the construction, fix a prime $p$, and let $A = \mathbb{F}_p[x_1, \ldots, x_n]$. Choose an absolute integral closure $A^+$ of $A$, i.e., $A^+$ is the normalization of $A$ in an algebraic closure of its fraction field. Recall the following theorem [HH92, §6.7]:

**Theorem 0.2** (Hochster-Huneke). The map $A \to A^+$ is flat.

To prove Proposition 0.1, it is enough to show:

**Proposition 0.3.** The $A$-algebra $A^+$ is not a filtered colimit of finitely presented flat $A$-algebras if $n \geq 3$.

**Proof.** We give an argument in the case $n = 3$, leaving the (obvious) generalization to the reader. It is enough to prove the analogous statement for the map $R \to R^+$, where $R$ is the strict henselization of $A$ at the origin (and is consequently a henselian regular local ring with residue field $\mathbb{F}_p$), and $R^+$ is its absolute integral closure.

Now choose an ordinary abelian surface $X$, $L$ with $X, L$ denote the preimage of $X, L$, and write $\Gamma_2$ for the induced finite surjective map. Since $X$ and $L$ are normal for $\mathbb{Z}$, there is a trace map $U \to \mathrm{Spec}(U)$ realizing a Noether normalization (see [Sta14, Tag 0571]). Part (b) was proven in [Bha12]; for the convenience of the reader, we recall the relevant argument.

Let $U \subset \mathrm{Spec}(S)$ be the punctured spectrum, so there are natural maps $X \leftarrow U \subset \mathrm{Spec}(S)$. The first map gives an identification $H^1(U, \mathcal{O}_U) \cong H^1(X, \mathcal{O}_X)$; by passing to the Witt vectors of the perfection and using the Artin-Schreier sequences, this gives an identification $H^1_{\et}(U, \mathbb{Z}_p) \cong H^1_{\et}(X, \mathbb{Z}_p)$. In particular, this group is a finite free $\mathbb{Z}_p$-module of rank 2 (since $X$ is ordinary). Now assume that there exists some $T$ as in (b) above. Let $V \subset \mathrm{Spec}(T)$ denote the preimage of $U$, and write $f : V \to U$ for the induced finite surjective map. Since $U$ is normal, there is a trace map $f_\ast \mathbb{Z}_p \to \mathbb{Z}_p$ on $U$ whose composition with the pullback $\mathbb{Z}_p \to f_\ast \mathbb{Z}_p$ is multiplication by $d = \deg(f)$. Passing to cohomology, and using that $H^1_{\et}(U, \mathbb{Z}_p)$ is non-torsion, then shows that $H^1_{\et}(V, \mathbb{Z}_p)$ is non-zero. Since $H^1_{\et}(V, \mathbb{Z}_p) \simeq \lim_{n\to\infty} H^1_{\et}(V, \mathbb{Z}/p^n)$ (as there is no $\lim_{n\to\infty}$ interference), the group $H^1(V_{\et}, \mathbb{Z}/p)$ must be non-zero. The Artin-Schreier sequence then shows $H^1(V, \mathcal{O}_V) \neq 0$. By excision, this gives $H^2_m(T) \neq 0$, where $m \in R$ is the maximal ideal. Thus, $T$ cannot be finite flat as an $R$-module since $H^2_m(R) = 0$, proving (b). □

REFERENCES

