TORSION DERIVED COMPLETIONS ARE SMALL

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The goal of this note is to record a purely algebraic consequence of the Banach open mapping theorem. Roughly speaking, it says that torsion complete modules tend to be small. More precisely, we have:

Theorem 0.1. *Fix a commutative ring A with a finitely generated ideal I. Let $K \in D(A)$ be a derived I-complete complex. Assume that $H^i(K)$ is acyclic outside $\mathrm{Spec}(A/I) \subset \mathrm{Spec}(A)$ for some index i. Then $H^i(K)$ is killed by I^n for some $n \geq 0$.*

Here a complex $K \in D(A)$ is said to derived I-complete[1] if for each $f \in I$, the limit

$$T(K, f) := R\lim(\ldots \xrightarrow{f} K \xrightarrow{f} K \xrightarrow{f} K) \in D(A)$$

vanishes. It is a basic fact that K is derived I-complete if and only if each $H^j(K)$ is so. Moreover, the collection of all derived I-complete A-modules forms an abelian (weak) Serre subcategory of all A-modules. In particular, the theorem above immediately reduces to the following lemma:

Lemma 0.2. *Fix a commutative ring A. Let $I = (t_1, \ldots, t_r)$ be a finitely generated ideal in A. Fix a derived I-complete A-module Q such that $Q[\frac{1}{t_i}] = 0$ for all i. Then Q is killed by I^n for some $n \geq 0$.*

Proof. By working with each generator of I separately, we may assume $I = (t)$. By restriction of scalars, we may then assume that $A = \mathbf{Z}[\![t]\!]$. Choose a presentation

$$F_1 \to F_0 \to Q \to 0$$

with F_i being free A-modules. Applying derived t-adic completions gives an exact sequence

$$\widehat{F_1} \xrightarrow{\phi} \widehat{F_0} \to Q \to 0$$

as Q is already derived t-complete (and because derived t-completion is right t-exact). Theorem 0.3 below shows that $\phi(\widehat{F_1})$ contains $t^n \widehat{F_0}$ for some $n \geq 0$, and thus, $t^n \cdot Q = 0$. □

The following form of the Banach Open Mapping Theorem was used above:

Theorem 0.3 (Banach)**.** *Let $A = \mathbf{Z}[\![t]\!]$. Let $\phi : P_1 \to P_0$ be a map between t-adic completions of free A-modules. Assume that $\mathrm{coker}(\phi)$ is t^∞-torsion. Then $\phi(P_1) \supset t^n P_0$ for some $n \geq 0$.*

This form of Banach's result can can be found in [He]. For convenience, we give an essentially complete proof below (modulo invoking the Baire category theorem).

Proof. The ring $A[\frac{1}{t}]$ is naturally a Banach algebra with unit ball given by A. Explicitly, for $f \in A[\frac{1}{t}]$, set

$$|f| = 2^{\mu_t(f)} \quad \text{where} \quad \mu_t(f) = \inf\{n \in \mathbf{Z} \mid t^n \cdot f \in A\}.$$

Similarly, for any free A-module P, there is an induced Banach $A[\frac{1}{t}]$-module structure on $\widehat{P}[\frac{1}{t}]$ with unit ball $\widehat{P}$. This construction endows $\widehat{P}[\frac{1}{t}]$ with a metric for which it is complete. Note that $t^n \widehat{P} \subset \widehat{P}[\frac{1}{t}]$ is a clopen subgroup, and gives a fundamental system of neighbourhoods of 0 as n varies.

By the preceding generalities, the $A[\frac{1}{t}]$-modules $P_i[\frac{1}{t}]$ in the theorem are topological $A[\frac{1}{t}]$-modules. Moreover, the map $\phi : P_1[\frac{1}{t}] \to P_0[\frac{1}{t}]$ is continuous (as $\phi^{-1}(t^n P_0) \supset t^n P_1$) and surjective (by assumption). Thus, we have

$$P_0[\frac{1}{t}] = \bigcup_n \phi(t^{-n} P_1).$$

[1]For the definition and basic properties of derived I-complete modules, see [SP, Tag 091N].

By the Baire category theorem, there exists some n such that the closure of $\phi(t^{-n}P_1)$ in $P_0[\frac{1}{t}]$ contains an open subset of $P_0[\frac{1}{t}]$. Thus, there exists some $z \in P_0[\frac{1}{t}]$ and some $k \geq 0$ such that

$$z + t^k P_0 \subset \overline{\phi(t^{-n}P_1)}.$$

It follows that for any $m \geq 0$, we have

$$z + t^k P_0 \subset \phi(t^{-n}P_1) + t^m P_0.$$

Taking $m = k+1$, and using the fact that the right side is closed under subtraction, we learn that

$$t^k P_0 \subset \phi(t^{-n}P_1) + t^{k+1}P_0.$$

Scaling by t^{-k} and renaming n, we have

$$P_0 \subset \phi(t^{-n}P_1) + tP_0.$$

Multiplying the t^m shows that

$$t^m P_0 \subset \phi(t^{m-n}P_1) + t^{m+1}P_0 \tag{1}$$

for any $m \geq 0$.

Now fix some element $y \in P_0$. We will check that $y = \phi(x)$ for some $x \in t^{-n}P_1$; this will prove that $P_0 \subset \phi(t^{-n}P_1)$, and thus $t^n P_0 \subset \phi(P_1)$, as wanted. Using (1) for $m = 0$, we can write

$$y = \phi(x_1) + y_1$$

with $x_1 \in t^{-n}P_1$, $y_1 \in tP_0$. Using (1) for $m = 1$ and y_1, we can write

$$y_1 = \phi(x_2) + y_2$$

with $x_2 \in t^{1-n}P_1$ and $y_2 \in t^2 P_0$. We can combine the two previous equalities to get

$$y = \phi(x_1 + x_2) + y_2$$

with $x_i \in t^{i-1-n}P_1$ and $y_i \in t^i P_0$. Continuing this way, we can inductively define $x_k \in t^{k-1-n}P_1$ and $y_k \in t^k P_0$ for all $k \geq 1$ such that

$$y = \phi(\sum_{k=1}^{N} x_k) + y_N$$

for all $N \geq 1$. The sequence $N \mapsto (\sum_{k=1}^{N} x_k)$ in $t^{-n}P_1$ is Cauchy, and thus converges to an element $x \in t^{-n}P_1$. The sequence $N \mapsto y_N$ in P_0 is Cauchy, and converges to 0. Thus, taking limits, we get

$$y = \phi(x),$$

as wanted. □

The proof above is an essentially analytic argument. We do not know a purely algebraic approach.

References

[SP] *The Stacks Project.* Available at http://stacks.math.columbia.edu.

[He] T. Henkel, *An Open Mapping Theorem for rings which have a zero sequence of units*, available at https://arxiv.org/pdf/1407.5647.pdf