REDUCING FLAT GROUPOIDS TO SMOOTH GROUPOIDS

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Fix a base scheme S, and let $\mathcal{X} \in \mathrm{Stacks}(\mathrm{Sch}(S)_{\mathrm{fppf}})$ be a sheaf of groupoids on the fppf site of S-schemes. Our goal is to explain a theorem of Artin that says that given an fppf atlas for $\mathcal{X}$, we can usually build a smooth atlas:

Theorem 1. *Let $\pi : X \to \mathcal{X}$ be a representable, separated, and fppf map with X an algebraic space locally of finite presentation over S. Assume that the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable. Then then there exists a representable, smooth, surjective, and separated map $\pi' : X' \to \mathcal{X}$ with X' an algebraic space locally of finite presentation over S.*

Example 2. Theorem 1 implies that for any affine fppf group scheme G over S, the stack $\mathbf{B}(G) = [S/G]$ of fppf G-torsors admits a smooth atlas. If G is smooth, then the quotient map $S \to \mathbf{B}(G)$ is a smooth atlas. If G is not smooth however, then the preceeding atlas is not smooth as the fibre is G. In this case, we may produce a smooth atlas by fixing an embedding $i : G \hookrightarrow \mathbf{GL}_n$ and using the natural map $\mathbf{GL}_n/G \to \mathbf{B}(G)$ as the atlas: the fibre is the smooth group scheme $\mathbf{GL}_n$. For example, for $G = \mu_p$ over $S = \mathbf{Z}_p$, this recipe applied to the tautological embedding $i : \mu_p \hookrightarrow \mathbf{G}_m$ produces the atlas $\mathbf{G}_m \to \mathbf{B}(\mu_p)$ determined by the fppf μ_p-torsor given by the p-power map $\mathbf{G}_m \to \mathbf{G}_m$.

The basic idea of the proof of Theorem 1 is to view π representing a space over $\mathcal{X}$ parametrising the subschemes of X which are sections of π, and then replace it by a space over $\mathcal{X}$ that parametrises local complete intersection (lci) subschemes of X which are multisections of π. As sections of smooth morphisms are lci, this modified space contains a component isomorphic to X if π is smooth. In general, the tradeoff presented by working with lci multisections instead of (possibly very singular) sections gives us a space with a well-behaved deformation theory. To make this idea precise, we use the following variant of the Hilbert scheme:

Definition 3. Given a representable map $X \to Y$ in $\mathrm{Stacks}(\mathrm{Sch}(S)_{\mathrm{fppf}})$, the functor $\mathrm{Hilb}_{\mathrm{lci}}(X/Y)$ on the category of Y-schemes is defined as follows:

$$\mathrm{Hilb}_{\mathrm{lci}}(X/Y)(T \to Y) = \begin{array}{l}\text{The set of all closed subschemes } i : Z \hookrightarrow X \times_Y T \\ \text{such that } Z \text{ is finite flat over } T\text{, and } i \text{ is a complete} \\ \text{intersection locally on } X_T.\end{array}$$

The main local property of $\mathrm{Hilb}_{\mathrm{lci}}(X/Y)$ we need is that, independent of any global hypothesis on X or Y, it's always represented by a smooth algebraic space covering Y:

Proposition 4. *Given a representable, separated, and fppf map $\pi : X \to Y$ in* $\mathrm{Stacks}(\mathrm{Sch}(S)_{\mathrm{fppf}})$*, the structure map $p : \mathrm{Hilb}_{\mathrm{lci}}(X/Y) \to Y$ is representable, smooth, surjective, and separated.*

Proof. To show representability, note that $\mathrm{Hilb}_{\mathrm{lci}}(X/Y)$ is naturally a subfunctor of $\mathrm{Hilb}(X/Y)$. As the latter is representable by Artin's theorems and the separatedness assumption, it suffices to show that the inclusion $\mathrm{Hilb}_{\mathrm{lci}}(X/Y) \subset \mathrm{Hilb}(X/Y)$ is an open subfunctor. Unwrapping the definitions, this boils to down to verifying the following: if (T,t) is a local Y-scheme, and $Z \subset X_T$ is a closed subscheme finite flat over T that is locally a complete intersection in the fibre X_t, then it is locally a complete intersection in all of X_T. As $Z \to T$ is finite and since there is nothing to show at points away from Z, it suffices to show that $Z \subset X_T$ is a complete intersection at the points of X_t. This follows from Grothendieck's flatness theorem which allows us to lift regular sequences across $\Gamma(X_T, \mathcal{O}) \to \Gamma(X_t, \mathcal{O})$ while specifying containment in the ideal of Z.

For the smoothness claim, fix an artinian local Y-scheme $\mathrm{Spec}(k)$, and a point $z : \mathrm{Spec}(k) \to \mathrm{Hilb}_{\mathrm{lci}}(X/Y)$ represented by a local complete intersection closed subscheme $i : Z \hookrightarrow X_k$ finite flat over $\mathrm{Spec}(k)$. We will verify that the deformations of z are unobstructed i.e., we will show that we can extend z across a given square zero thickening $\mathrm{Spec}(k) \subset \mathrm{Spec}(k')$ in the category of Y-schemes. As $|Z|$ is a finite set of distict closed points, we may work with one point at a time and, consequently, assume that $|Z|$ is reduced to a point $\{z\}$. Let S denote the local ring of $X_{k'}$ at z, and let $I \subset k'$ be the ideal defining k. By assumption, there exists a regular sequence $\bar{\mathbf{f}} = (\bar{f}_1, \ldots, \bar{f}_n)$ in $\mathcal{O}_{X_k,z} = S/IS$ defining Z. By Grothendieck's flatness theorem, there exists a regular sequence $\mathbf{f} = (f_1, \ldots, f_n)$ in S reducing to $\bar{\mathbf{f}}$. Moreover, the same theorem also ensures that $S/(\mathbf{f})$ is flat over R. Nakayama's lemma implies that $S/(\mathbf{f})$ is finite over R. By spreading out, we see that $\mathbf{f}$ is a regular sequence around $z \in X_{k'}$ defining a local complete intersection subscheme $Z' \subset X_{k'}$ such that $Z' \times_k k' = Z$, and Z is finite flat over k'. This verifies that the deformations of z are unobstructed.

To show surjectivity, it suffices to show that for a field k, every finite type k-scheme X admits an lci closed subscheme of dimension 0. Clearly it suffices to show this for X affine. If $\dim(X) = 0$, there is nothing to show. By induction, it suffices to show that $A = \Gamma(X, \mathcal{O})$ admits a regular element. This follows from the infinitude of set of all primes of A coming from the assumption $\dim(X) > 0$, the finiteness of the set of associated primes of A as A is noetherian, and prime avoidance.

The separatedness of p can be checked using the valuative criterion and the fact the diagonal $\Delta : X \to X \times_Y X$ is a closed immersion (the separatedness of π) as follows: given a Y-trait T with generic point η, two finite flat covers $p_1 : Z_1 \to T$ and $p_1 : Z_2 \to T$ with an isomorphism $(Z_1)_\eta \simeq (Z_2)_\eta$, and two Y-closed immersions $i_1 : Z_1 \hookrightarrow X$ and $i_2 : Z_2 \hookrightarrow X$ which agree on $(Z_1)_\eta \simeq (Z_2)_\eta$, the closed subscheme $i_1 \times_Y i_2 : Z_1 \times_Y Z_2 \hookrightarrow X \times_Y X$ has a dense open subscheme $(Z_1 \times_Y Z_2)_\eta$ contained in the diagonal $\Delta(X)$. As the latter is closed, it follows that $Z_1 \times_Y Z_2 \subset \Delta(X)$, as desired. ☐

Proposition 4 gives us access to an atlas with good relative properties. In order to prove to build an atlas with good properties globally, we need an auxilliary object.

Definition 5. For an S-scheme W, define the stack $\mathrm{Cov}(W)$ on $\mathrm{Sch}(S)$ as follows:

$$\mathrm{Cov}(W)(T) = \begin{array}{l}\text{The groupoid of pairs } (p : Z \to T, f : Z \to W) \text{ where } p \text{ is a finite flat cover, and } f \\ \text{is some map.}\end{array}$$

The only result we need concerning $\mathrm{Cov}(W)$ is its algebraicity.

Proposition 6. *If W is an S-scheme locally of finite presentation, then* $\mathrm{Cov}(W)$ *is a locally algebraic stack on S.*

Proof. The sheaf condition is easy to verify. In order to verify the algebraicity, consider first the case $S = W$. In this case, $\mathrm{Cov}(S)(T)$ is simply the groupoid of finite flat covers of T. Fixing a relative basis produces a smooth presentation for $\mathrm{Cov}(S)$. Thus, $\mathrm{Cov}(S)$ is locally algebraic. For a general W, it now suffices to show that $\mathrm{Cov}(W) \to \mathrm{Cov}(S)$ is representable. The fibre of this map over a point $T \to \mathrm{Cov}(S)$ is the sheaf $p_*(W \times_S Z)$ where $p : Z \to T$ is cover corresponding to the chosen point $T \to \mathrm{Cov}(S)$. As the map p is finite flat, the claim now follows from the general fact that the restriction of scalars of a representable sheaf remains representable under a finite flat map. ☐

We now have enough tools to finish the proof of the main theorem.

Proof of Theorem 1. Assume for a moment that Theorem 1 is proven when $\mathcal{X}$ is discrete i.e., a sheaf of sets. Then, for a sheaf of groupoids $\mathcal{X}$ with a map $\pi : X \to \mathcal{X}$ as in the statement of the theorem, we may form the diagram

$$\begin{array}{ccc} \mathrm{Hilb}_{\mathrm{lci}}(X/\mathcal{X}) \times_{\mathcal{X}} X & \xrightarrow{\mathrm{pr}_1} & \mathrm{Hilb}_{\mathrm{lci}}(X/\mathcal{X}) \\ \downarrow{\scriptstyle \mathrm{pr}_2} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{\pi} & \mathcal{X} \end{array}$$

where p is the structure morphism. As X is discrete as a sheaf on S-schemes, so is the sheaf $\mathrm{Hilb}_{\mathrm{lci}}(X/\mathcal{X})$. By Proposition 4, the map p is smooth, surjective and separated. It therefore suffices to show that $\mathrm{Hilb}_{\mathrm{lci}}(X/\mathcal{X})$ is an algebraic space. As the structure map p is separated and $\mathcal{X}$ has a representable diagonal, it follows that $\mathrm{Hilb}_{\mathrm{lci}}(X/\mathcal{X})$ also has a representable diagonal. On the other hand, the map pr_1 is a representable, separated, and fppf being the base change of such a map. By the assumption that Theorem 1 is known for discrete sheaves, it now suffices to verify that $\mathrm{Hilb}_{\mathrm{lci}}(X/\mathcal{X}) \times_{\mathcal{X}} X$ is representable. This follows from the representability of X and pr_2.

It remains to verify Theorem 1 under the additional assumption that $\mathcal{X}$ is discrete. As the map $p : \mathrm{Hilb}_{\mathrm{lci}}(X/\mathcal{X}) \to \mathcal{X}$ is representable, smooth, surjective, and separated by Proposition 4, it suffices to verify that $\mathrm{Hilb}_{\mathrm{lci}}(X/\mathcal{X})$ is representable. Thanks to Proposition 6, it suffices to exhibit $\mathrm{Hilb}_{\mathrm{lci}}(X/\mathcal{X})$ as a representable substack of $\mathrm{Cov}(X)$. When viewed as a stack on $\mathrm{Sch}(S)$, a T-point of $\mathrm{Hilb}_{\mathrm{lci}}(X/\mathcal{X})$ is given by the data of a map $T \to \mathcal{X}$, a finite flat cover $Z \to T$, and a $\mathcal{X}$-map $f : Z \to X$ such that the resulting map $i : Z \to X \times_{\mathcal{X}} T$ is an lci closed immersion. Forgetting the map $T \to Y$, we obtain a morphism $i : \mathrm{Hilb}_{\mathrm{lci}}(X/\mathcal{X}) \to \mathrm{Cov}(X)$. As $\mathcal{X}$ is discrete, the map $T \to \mathcal{X}$ is uniquely determined by the map $Z \to X \to \mathcal{X}$ if it exists. Thus, $\mathrm{Hilb}_{\mathrm{lci}}(X/\mathcal{X})$ is naturally a subfunctor of $\mathrm{Cov}(X)$. To show that it is a representable subfunctor, note that the essential image of i can be characterised as follows:

$$i(\mathrm{Hilb}_{\mathrm{lci}}(X/\mathcal{X}))(T) = \begin{array}{l}\text{The groupoid of pairs } (p : Z \to T, f : Z \to X) \in \mathrm{Cov}(X)(T) \\ \text{such that the induced map } Z \times_T Z \to X \times_S X \text{ factors} \\ \text{through } X \times_{\mathcal{X}} X \hookrightarrow X \times_S X\end{array}$$

The existence of such a characterisation follows from the fact that $\mathcal{X}$ is a colimit of the diagram

$$X \times_{\mathcal{X}} X \rightrightarrows X$$

On the other hand, the assumption on the diagonal of $\mathcal{X}$ and the cartesian square

$$\begin{array}{ccc} X \times_{\mathcal{X}} X & \longrightarrow & X \times_S X \\ \downarrow & & \downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{X} \times_S \mathcal{X} \end{array}$$

imply that the map $X \times_{\mathcal{X}} X \hookrightarrow X \times_S X$ is a representable. It follows from the preceeding characterisation that i is representable, thereby establishing the claim. □