REDDUCING FLAT GROUPOIDS TO SMOOTH GROUPOIDS

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Fix a base scheme $S$, and let $\mathcal{X} \in \text{Stacks}(\text{Sch}(S)_{\text{fppf}})$ be a sheaf of groupoids on the fppf site of $S$-schemes. Our goal is to explain a theorem of Artin that says that given an fppf atlas for $\mathcal{X}$, we can usually build a smooth atlas:

**Theorem 1.** Let $\pi : X \to \mathcal{X}$ be a representable, separated, and fppf map with $X$ an algebraic space locally of finite presentation over $S$. Assume that the diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable. Then there exists a representable, smooth, surjective, and separated map $\pi' : X' \to X$ with $X'$ an algebraic space locally of finite presentation over $S$.

**Example 2.** Theorem 1 implies that for any affine fppf group scheme $G$ over $S$, the stack $\mathcal{B}(G) = [S/G]$ of fppf $G$-torsors admits a smooth atlas. If $G$ is smooth, then the quotient map $S \to \mathcal{B}(G)$ is a smooth atlas. If $G$ is not smooth however, then the preceeding atlas is not smooth as the fibre is $G$. In this case, we may produce a smooth atlas by fixing an embedding $i : G \to \text{GL}_n$ and using the natural map $\text{GL}_n/G \to \mathcal{B}(G)$ as the atlas: the fibre is the smooth group scheme $\text{GL}_n$. For example, for $G = \mu_p$ over $S = \mathbb{Z}_{p^n}$, this recipe applied to the tautological embedding $i : \mu_p \to \mathbb{G}_m$ produces the atlas $\mathbb{G}_m \to \mathbb{B}(\mu_p)$ determined by the fppf $\mu_p$-torsors given by the $p$-power map $\mathbb{G}_m \to \mathbb{G}_m$.

The basic idea of the proof of Theorem 1 is to view $\pi$ representing a space over $\mathcal{X}$ parametrising the subschemes of $X$ which are sections of $\pi$, and then replace it by a space over $\mathcal{X}$ that parametrises local complete intersection (lci) subschemes of $X$ which are multisections of $\pi$. As sections of smooth morphisms are lci, this modified space contains a component isomorphic to $X$ if $\pi$ is smooth. In general, the tradeoff presented by working with lci multisections instead of (possibly very singular) sections gives us a space with a well-behaved deformation theory. To make this idea precise, we use the following variant of the Hilbert scheme:

**Definition 3.** Given a representable map $X \to Y$ in $\text{Stacks}(\text{Sch}(S)_{\text{fppf}})$, the functor $\text{Hilb}_{\text{lci}}(X/Y)$ on the category of $Y$-schemes is defined as follows:

$$\text{Hilb}_{\text{lci}}(X/Y)(T \to Y) = \text{such that } Z \text{ is finite flat over } T, \text{ and } i \text{ is a complete intersection locally on } X_T.$$

The main local property of $\text{Hilb}_{\text{lci}}(X/Y)$ we need is that, independent of any global hypothesis on $X$ or $Y$, it's always represented by a smooth algebraic space covering $Y$:

**Proposition 4.** Given a representable, separated, and fppf map $\pi : X \to Y$ in $\text{Stacks}(\text{Sch}(S)_{\text{fppf}})$, the structure map $p : \text{Hilb}_{\text{lci}}(X/Y) \to Y$ is representable, smooth, surjective, and separated.

**Proof.** To show representability, note that $\text{Hilb}_{\text{lci}}(X/Y)$ is naturally a subfunctor of $\text{Hilb}(X/Y)$. As the latter is representable by Artin’s theorems and the separatedness assumption, it suffices to show that the inclusion $\text{Hilb}_{\text{lci}}(X/Y) \subset \text{Hilb}(X/Y)$ is an open subfunctor. Unwrapping the definitions, this boils down to verifying the following: if $(T,i)$ is a local $Y$-scheme, and $Z \subset X_T$ is a closed subscheme finite flat over $T$ that is locally a complete intersection in the fibre $X_t$, then it is locally a complete intersection in all of $X_T$. As $Z \to T$ is finite and since there is nothing to show at points away from $Z$, it suffices to show that $Z \subset X_T$ is a complete intersection at the points of $X_t$. This follows from Grothendieck’s flatness theorem which allows us to lift regular sequences across $\Gamma(X_T, \mathcal{O}) \to \Gamma(X, \mathcal{O})$ while specifying containment in the ideal of $Z$.

For the smoothness claim, fix an artinian local $Y$-scheme $\text{Spec}(k)$, and a point $z : \text{Spec}(k) \to \text{Hilb}_{\text{lci}}(X/Y)$ represented by a local complete intersection closed subscheme $i : Z \to X_k$ finite flat over $\text{Spec}(k)$. We will verify that the deformations of $z$ are unobstructed i.e., we will show that we can extend $z$ across a given square zero thickening $\text{Spec}(k) \subset \text{Spec}(k')$ in the category of $Y$-schemes. As $|Z|$ is a finite set of distinct closed points, we may work with one point at a time and, consequently, assume that $|Z|$ is reduced to a point $\{z\}$. Let $S$ denote the local ring of $X_{k'}$ at $z$, and let $I \subset k'$ be the ideal defining $k$. By assumption, there exists a regular sequence $f = (f_1, \ldots, f_n)$ in $\mathcal{O}_{X_k,z} = S/IS$ defining $Z$. By Grothendieck’s flatness theorem, there exists a regular sequence $\mathcal{f} = (f_1, \ldots, f_n)$ in $S$ reducing to $\mathcal{f}$. Moreover, the same theorem also ensures that $S/(\mathcal{f})$ is flat over $\mathcal{R}$. Nakayama’s lemma implies that $S/(\mathcal{f})$ is finite.
over $R$. By spreading out, we see that $f$ is a regular sequence around $z \in X_0$, defining a local complete intersection subscheme $Z' \subset X_0$ such that $Z' \times_k k' = Z$, and $Z$ is finite flat over $k'$. This verifies that the deformations of $z$ are unobstructed.

To show surjectivity, it suffices to show that for a field $k$, every finite type $k$-scheme $X$ admits an lci closed subscheme of dimension 0. Clearly it suffices to show this for $X$ affine. If $\dim(X) = 0$, there is nothing to show. By induction, it suffices to show that $A = \Gamma(X, \mathcal{O})$ admits a regular element. This follows from the infinitude of set of all primes of $A$ coming from the assumption $\dim(X) > 0$, the finiteness of the set of associated primes of $A$ as $A$ is noetherian, and prime avoidance.

The separatedness of $p$ can be checked using the valuative criterion and the fact the diagonal $\Delta : X \rightarrow X \times_Y X$ is a closed immersion (the separatedness of $\pi$) as follows: given a $Y$-trait $T$ with generic point $\eta$, two finite flat covers $p_1 : Z_1 \rightarrow T$ and $p_2 : Z_2 \rightarrow T$ with an isomorphism $(Z_1)_\eta \simeq (Z_2)_\eta$, and two $Y$-closed immersions $i_1 : Z_1 \hookrightarrow X$ and $i_2 : Z_2 \hookrightarrow X$ which agree on $(Z_1)_\eta \simeq (Z_2)_\eta$, the closed subscheme $i_1 \times_Y i_2 : (Z_1 \times_Y Z_2)_\eta$ contained in the diagonal $\Delta(X)$. As the latter is closed, it follows that $Z_1 \times_Y Z_2 \subset \Delta(X)$, as desired.

Proposition 4 gives us access to an atlas with good relative properties. In order to prove to build an atlas with good properties globally, we need an auxiliary object.

**Definition 5.** For an $S$-scheme $W$, define the stack $\text{Cov}(W)$ on $\text{Sch}(S)$ as follows:

$$\text{Cov}(W)(T) = \{ \text{the groupoid of pairs } (p : Z \rightarrow T, f : Z \rightarrow W) \text{ where } p \text{ is a finite flat cover, and } f \text{ is some map. } \}$$

The only result we need concerning $\text{Cov}(W)$ is its algebraicity.

**Proposition 6.** If $W$ is an $S$-scheme locally of finite presentation, then $\text{Cov}(W)$ is a locally algebraic stack on $S$.

**Proof.** The sheaf condition is easy to verify. In order to verify the algebraicity, consider first the case $S = W$. In this case, $\text{Cov}(S)(T)$ is simply the groupoid of finite flat covers of $T$. Fixing a relative basis produces a smooth presentation for $\text{Cov}(S)$. Thus, $\text{Cov}(S)$ is locally algebraic. For a general $W$, it now suffices to show that $\text{Cov}(W) \rightarrow \text{Cov}(S)$ is representable. The fibre of this map over a point $T \rightarrow \text{Cov}(S)$ is the sheaf $p_!(W \times_Z T)$ where $p : Z \rightarrow T$ is cover corresponding to the chosen point $T \rightarrow \text{Cov}(S)$. As the map $p$ is finite flat, the claim now follows from the general fact that the restriction of scalars of a representable sheaf remains representable under a finite flat map. □

We now have enough tools to finish the proof of the main theorem.

**Proof of Theorem 1.** Assume for a moment that Theorem 1 is proven when $\mathcal{X}$ is discrete i.e., a sheaf of sets. Then, for a sheaf of groupoids $\mathcal{X}$ with a map $\pi : X \rightarrow \mathcal{X}$ as in the statement of the theorem, we may form the diagram

$$\begin{array}{ccc}
\text{Hilb}_{lci}(X/\mathcal{X}) \times_{\mathcal{X}} X & \xrightarrow{p_1} & \text{Hilb}_{lci}(X/\mathcal{X}) \\
\downarrow{p_2} & & \downarrow{p} \\
X & \xrightarrow{\pi} & \mathcal{X}
\end{array}$$

where $p$ is the structure morphism. As $X$ is discrete as a sheaf on $S$-schemes, so is the sheaf $\text{Hilb}_{lci}(X/\mathcal{X})$. By Proposition 4, the map $p$ is smooth, surjective and separated. It therefore suffices to show that $\text{Hilb}_{lci}(X/\mathcal{X})$ is an algebraic space. As the structure map $p$ is separated and $\mathcal{X}$ has a representable diagonal, it follows that $\text{Hilb}_{lci}(X/\mathcal{X})$ also has a representable diagonal. On the other hand, the map $p_1$ is a representable, separated, and fppf being the base change of such a map. By the assumption that Theorem 1 is known for discrete sheaves, it now suffices to verify that $\text{Hilb}_{lci}(X/\mathcal{X}) \times_{\mathcal{X}} X$ is representable. This follows from the representability of $X$ and $p_2$.

It remains to verify Theorem 1 under the additional assumption that $\mathcal{X}$ is discrete. As the map $p : \text{Hilb}_{lci}(X/\mathcal{X}) \rightarrow \mathcal{X}$ is representable, smooth, surjective, and separated by Proposition 4, it suffices to verify that $\text{Hilb}_{lci}(X/\mathcal{X})$ is representable. Thanks to Proposition 6, it suffices to exhibit $\text{Hilb}_{lci}(X/\mathcal{X})$ as a representable substack of $\text{Cov}(X)$. When viewed as a stack on $\text{Sch}(S)$, a $T$-point of $\text{Hilb}_{lci}(X/\mathcal{X})$ is given by the data of a map $T \rightarrow \mathcal{X}$, a finite flat cover $Z \rightarrow T$, and an $X$-map $f : Z \rightarrow X$ such that the resulting map $i : Z \rightarrow X \times_{\mathcal{X}} T$ is an lci closed immersion. Forgetting the map $T \rightarrow Y$, we obtain a morphism $i : \text{Hilb}_{lci}(X/\mathcal{X}) \rightarrow \text{Cov}(X)$. As $\mathcal{X}$ is discrete, the map $T \rightarrow \mathcal{X}$ is uniquely determined.
by the map \( Z \to X \to \mathcal{X} \) if it exists. Thus, \( \text{Hilb}_{\text{lc}}(X/\mathcal{X}) \) is naturally a subfunctor of \( \text{Cov}(X) \). To show that it is a representable subfunctor, note that the essential image of \( i \) can be characterised as follows:

\[
i(\text{Hilb}_{\text{lc}}(X/\mathcal{X}))(T) = \text{The groupoid of pairs } (p : Z \to T, f : Z \to X) \in \text{Cov}(X)(T)
\]

The groupoid of pairs \( (p : Z \to T, f : Z \to X) \in \text{Cov}(X)(T) \)

such that the induced map \( Z \times_T Z \to X \times_S X \) factors through \( X \times_{\mathcal{X}} X \to X \times_S X \).

The existence of such a characterisation follows from the fact that \( \mathcal{X} \) is a colimit of the diagram

\[
\begin{array}{ccc}
X \times_{\mathcal{X}} X & \to & X \\
\downarrow & & \downarrow \\
\mathcal{X} & \to & X \times_S X
\end{array}
\]

On the other hand, the assumption on the diagonal of \( \mathcal{X} \) and the cartesian square

\[
\begin{array}{ccc}
X \times_{\mathcal{X}} X & \to & X \times_S X \\
\downarrow & & \downarrow \\
\mathcal{X} & \to & X \times_S X
\end{array}
\]

imply that the map \( X \times_{\mathcal{X}} X \to X \times_S X \) is a representable. It follows from the preceding characterisation that \( i \) is representable, thereby establishing the claim. \( \square \)