REFINED ALTERATIONS

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1. INTRODUCTION

Fix an algebraic variety X over a field k. Hironaka showed [Hir64] that if k has characteristic 0, then one can find a proper birational map $\pi : Y \to X$ with Y smooth. de Jong showed [dJ96] the same is true in arbitrary characteristic if we replace "birational" with "generically finite." Hironaka's theorem also shows that π can be chosen to be an isomorphism over X^{sm}. On the other hand, it is sometimes asserted that de Jong's method does not allow for any control on the locus where π is well-behaved. The goal of this note is to show that, in fact, de Jong's alterations theorem can be proven while exercising some control over the étale locus of the alteration:

Theorem 1.1. *Let X be a variety over a perfect field k, and let $F \subset X^{sm}$ be a finite subset of closed points the smooth locus. Then there exists an alteration $\pi : Y \to X$ such that Y is smooth and π is finite étale in a neighbourhood of F.*

As in Hironaka's case, a stronger variant of Theorem 1.1 involving pairs (see Theorem 2.6) is also true. An important corollary of Hironaka's *construction* is the existence of good compactifications in characteristic 0. Using de Jong's methods, we obtain good compactifications in any characteristic after passage to an étale cover:

Theorem 1.2. *Let U be a smooth variety over a perfect field k. Then there exists an étale cover $V \to U$, and a dense open immersion $V \subset \overline{V}$ which is the complement of a strict normal crossings divisor in a smooth projective variety $\overline{V}$.*

A fundamental consequence of Hironaka's theorem and the theory of toroidal embeddings is the semistable reduction theorem in characteristic 0. Using de Jong's method, we prove the same in mixed characteristic up to passage to an étale cover; here a local field is a finite extension of $W(k)[1/p]$ with k a perfect characteristic p field.

Theorem 1.3. *Let U be a smooth variety over a local field K. Then there exists a finite extension K'/K, an étale cover $V \to U$, and a dense open immersion $V \subset \overline{V}$ which is the complement of a strict normal crossings divisor in a proper semistable $\mathcal{O}_{K'}$-scheme $\overline{V}$.*

Similar results in the rigid analytic category were proven in [Har03] and [Fal88, §III.2]; both these results require one to work with genuine analytic spaces even when starting with schemes, and hence do not recover Theorem 1.3.

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2. THE EQUICHARACTERISTIC SETTING

Fix a perfect field k. Our goal in this section formulate a statement that implies Theorems 1.1 and 1.2; this stronger statement is more robust for inducative arguments, and is formulated in terms of the following category of pairs inspired by [Bei12, §2]:

Definition 2.1. We define the category $\mathcal{P}_k$ of *pairs* as follows. The objects are (scheme-theoretically) dense open immersions $j : U \hookrightarrow \overline{U}$ where U is a smooth quasi-projective k-variety, and $\overline{U}$ is proper k-variety, and a map is given by the obvious commutative square. We may abusively write objects of $\mathcal{P}_k$ as $(U, \overline{U})$. For a map $g : (U, \overline{U}) \to (V, \overline{V})$ in $\mathcal{P}_k$, we often write $g : U \to V$ and $\overline{g} : \overline{U} \to \overline{V}$ for the induced maps. We say that a map $g : (U, \overline{U}) \to (V, \overline{V})$ is *étale* (resp. *finite étale*) if the map $U \to V$ is so. Given a finite subscheme $Z \subset V$, we say that an étale map $g : (U, \overline{U}) \to (V, \overline{V})$ is a *finite étale neighbourhood* of Z if the map $U \to V$ is finite étale in a neighbourhood of Z. Let $\mathcal{P}_k^{\mathrm{ss}} \subset \mathcal{P}_k$ be the full subcategory of *semistable pairs*, i.e., pairs $(U, \overline{U})$ where $\overline{U}$ is smooth and projective, and $\overline{U} - U$ is a strict normal crossings divisor. A family $\{f_i : (U_i, \overline{U_i}) \to (V, \overline{V})\}$ of maps in $\mathcal{P}_k$ or $\mathcal{P}_K^{\mathrm{ss}}$ is called an *étale cover* if each f_i is étale, and $\cup_i f_i(U_i) = U$.

Remark 2.2. Given a pair $(U, \overline{U})$, all generic points of $\overline{U}$ lie in U by the scheme-theoretic density. In particular, $\overline{U}$ is reduced. If moreover $\overline{U}$ is normal, then $\pi_0(\overline{U}) = \pi_0(U)$. Moreover, an étale map $f : (U, \overline{U}) \to (V, \overline{V})$ of pairs is precisely a proper surjective map $\overline{U} \to \overline{V}$ which carries U to V and is étale over U.

Example 2.3. A basic example of an étale map in $\mathcal{P}_k$ is $(U, \overline{U}^{\mathrm{norm}}) \to (U, \overline{U})$ where $\overline{U}^{\mathrm{norm}}$ denotes the normalisation of $\overline{U}$ in U; this map is a finite étale neighbourhood of any finite subscheme of U. More generally, any U-admissible modification of $\overline{U}$, i.e., a proper birational map $\overline{V} \to \overline{U}$ which is an isomorphism over U, provides such an example.

Remark 2.4. The category $\mathcal{P}_k$ admits certain limits. For example, if $f : (X, \overline{X}) \to (Y, \overline{Y})$ is any map in $\mathcal{P}_k$ and $\epsilon : (Y', \overline{Y'}) \to (Y, \overline{Y})$ is an étale map, then the base change of f along ϵ exists, is étale over $(X, \overline{X})$, and is given by $(X', \overline{X'})$ where $X' = X \times_Y Y'$ and $\overline{X'}$ is the scheme-theoretic closure of X' in $\overline{X} \times_{\overline{Y}} \overline{Y'}$. Given a finite subscheme $F_X \subset X$, if $\epsilon : Y' \to \overline{Y}$ is a finite étale neighbourhood of $f(F_X)$, then $(X', \overline{X'}) \to (X, \overline{X})$ is a finite étale neighbourhood of F_X. The induced map $(X', \overline{X'}^{\mathrm{norm}}) \to (Y', \overline{Y'})$ is called the *normalised* base change of π along ϵ.

The main theorem of this note is the following:

Theorem 2.5. *Let $(X, \overline{X}) \in \mathcal{P}_k$, and fix a finite reduced subscheme $F_X \subset X$. Then there exists a finite étale neighbourhood $\pi : (U, \overline{U}) \to (X, \overline{X})$ of F_X with $(U, \overline{U})$ semistable.*

Theorem 2.5 immediately implies Theorem 1.1 as well as Theorem 1.2. In fact, since a subdivisor of a strict normal crossings divisor is also a strict normal crossings divisor, we obtain the following analogue of [dJ96, Theorem 4.1]:

Theorem 2.6. *Let k be a perfect field. Let X be a variety over k. Fix a closed subset $Z \subset X$ and a finite reduced subscheme $F_X \subset (X - Z)^{\mathrm{sm}}$. Then there exists an alteration $\phi_1 : X_1 \to X$ such that*

(1) X_1 is a smooth projective k-variety.
(2) $\phi_1^{-1}(Z)$ is a strict normal crossings divisor in X_1.
(3) ϕ_1 is finite étale over F_X.

Theorem 2.5 also has consequences for the étale topology. Let Sm_k denote the category of smooth k-varieties equipped with the étale topology. Using the definition of étale covers given in Definition 2.1, one obtains:

Theorem 2.7. *Restriction induces equivalences of topoi:* $\mathrm{Shv}_{\text{ét}}(\mathcal{P}_k) \simeq \mathrm{Shv}_{\text{ét}}(\mathcal{P}_k^{\mathrm{ss}}) \simeq \mathrm{Shv}_{\text{ét}}(\mathrm{Sm}_k)$.

Proof. Consider the functors $\mathcal{P}_k^{\mathrm{ss}} \xrightarrow{i} \mathcal{P}_k \xrightarrow{r} \mathrm{Sm}_k$ where i is the inclusion, and r is given by $r(U, \overline{U}) = U$. For any object $(U, \overline{U}) \in \mathcal{P}_k$, Theorem 2.5 gives an étale cover $\{(U_i, \overline{U_i}) \to (U, \overline{U})\}$ with $(U_i, \overline{U_i}) \in \mathcal{P}_k^{\mathrm{ss}}$. A simple argument using Theorem 2.5 shows that any map in $\mathcal{P}_k$ also comes from a map in $\mathcal{P}_k^{\mathrm{ss}}$, at least after replacing the source and target by étale covers. This proves that i induces an equivalence by [Bei12, §2.1]. The claim for r is similar: the functor r is faithful, any $U \in \mathrm{Sm}_k$ compactifies to a pair $(U, \overline{U}) \in \mathcal{P}_k$ as U is quasi-projective, and maps $U \to V$ in Sm_k extend to maps $(U, \overline{U}) \to (V, \overline{V})$ in $\mathcal{P}_k$, at least after a U-admissible blowup of $\overline{U}$. ☐

Remark 2.8. We had originally hoped that Theorem 2.7 would give a definition of the log crystalline cohomology of a smooth variety without assuming that good compactifications exist. However, this does not seem to be possible: the presheaf $(U, \overline{U}) \mapsto \mathrm{R}\Gamma_{\mathrm{crys}}((U, \overline{U}), \mathcal{O})$ of crystalline cohomology complexes of the logarithmic scheme $(U, \overline{U})$ relative to $\mathbf{Z}/p^n$ (for some n) does not satisfy étale descent on $\mathcal{P}_k^{\mathrm{ss}}$, and its étale sheafification is too large, even for $(\mathbf{A}^1, \mathbf{P}^1)$. Indeed, even for $n = 1$, one has $\mathrm{R}\Gamma_{\mathrm{crys}}((\mathbf{A}^1, \mathbf{P}^1), \mathcal{O}) \simeq k$, but it is easy to check that this value does not satisfy cohomological descent for the Artin-Schreier cover $(\mathbf{A}^1, \mathbf{P}^1) \to (\mathbf{A}^1, \mathbf{P}^1)$.

3. Proof of Theorem 2.5

Fix a pair $(X, \overline{X})$ and a finite reduced subscheme $F_X \subset X$ as in Theorem 2.5. Write $d = \dim(X)$.

3.1. **Strategy.** Assume we have constructed a finite étale neighbourhood $g : (X', \overline{X'}) \to (X, \overline{X})$ of F_X. Then to prove Theorem 2.5 for the triple $((X, \overline{X}), F_X)$, it suffices to prove the same for $((X', \overline{X'}), g^{-1}F_X)$. The goal of the rest of this section is to construct a sequence of such neighbourhoods g which progressively improve the singularities of the pair $(X, \overline{X})$. In practice, these will typically be produced as follows: we will construct a curve fibration $\pi : (X, \overline{X}) \to (Y, \overline{Y})$, a finite étale neighbourhood $\epsilon : (Y', \overline{Y'}) \to (Y, \overline{Y})$ of $F_Y := \pi(F_X) \subset Y$, and set g to be the map from the normalised base change of π along ϵ to $(X, \overline{X})$.

3.2. **Preliminary reductions.** Clearly we may assume that $\overline{X}$ is connected. By limit arguments, we may assume $k = \overline{k}$. Since X is quasi-projective, it admits a compactification X'. The normalised closure of the graph the induced rational map from $\overline{X}$ to X' defines proper birational X-admissible maps $\overline{X} \xleftarrow{a} X'' \xrightarrow{b} X'$ with X'' normal and proper. By Raynaud-Gruson [RG71, Theorem 5.2.2], blowups are cofinal amongst all X-admissible modifications of X'. Hence, after replacing X'' with a further X-admissible modification, one may assume that b is a blowup. In particular, both b and X'' are projective. Replacing $(X, \overline{X})$ with (X, X''), we may assume that $\overline{X}$ is normal and projective.

3.3. **Finding projections:** [dJ96, §2.11,4.11, 4.12]**.** Let $((X, \overline{X}), F_X)$ be as in §3.2. Our goal is to construct a curve fibration π on $\overline{X}$ that has the following two properties: (a) at least after shrinking around F_X, π realises an elementary fibration over X (in the sense of [SGA73, Expose XI]), and (b) F_X lives in a fibre of π.

Proposition. *After replacing $(X, \overline{X})$ with a finite étale neighbourhood of F_X, we may assume that there exists a map $\pi : (X, \overline{X}) \to (Y, \overline{Y})$ satisfying:*

(1) $\overline{X}$ and $\overline{Y}$ are normal projective k-varieties.
(2) The map $\overline{\pi}$ is a proper generically smooth map, and the smooth locus $(\overline{X}/\overline{Y})^{\mathrm{sm}}$ is dense in all fibres.
(3) All fibres of of $\overline{\pi}$ are non-empty, geometrically connected, and of equidimension 1.
(4) The map $\overline{\pi}$ is smooth over Y.
(5) The open subscheme $X \subset \overline{X} \times_{\overline{Y}} Y$ is the complement of a divisor finite étale over Y.
(6) The subscheme $F_X \subset X$ lies in a single smooth fibre of $\overline{\pi}$.

The key is the following.

Lemma. *Let $W \subset \mathbf{P}^N$ be a subvariety of dimension $d-2$, and let $F \subset \mathbf{P}^N(k)$ be a finite set of rational points not on W. Then, after possibly replacing $\mathbf{P}^N$ with the target of a Veronese embedding, the space of codimension $(d-1)$-planes $L \subset \mathbf{P}^N$ that pass through all points of F and miss W completely forms a non-empty Zariski open dense subset of the set of all codimension $(d-1)$-planes L that pass through all points of F.*

Proof. After possibly applying a large enough Veronese embedding, we may assume: (a) the points in F are in linearly general position, and (b) all points on W are in linearly general position with respect to points in F. The space G of codimension $(d-1)$-planes in $\mathbf{P}^N$ has dimension $(d-1)(N+1)-(d-1)^2$. The subspace $G_F \subset G$ of planes passing through all points of F has dimension $(d-1)(N+1-r)-(d-1)^2$, where $r = \#F$. For each point $z \in W(k)$, the subspace $G_{F \cup z} \subset G_F$ of codimension $(d-1)$-planes in $\mathbf{P}^N$ that go through all points of F and z has dimension $(d-1)(N-r)-(d-1)^2$. Hence, the subspace $G_{F,W}$ of all codimension $(d-1)$-planes in $\mathbf{P}^N$ that go through all points of F and some point of W has dimension $\leq (d-1)(N+r)-(d-1)^2+(d-2) < \dim(G_F)$, which shows that $G_{F,W}$ is a proper subspace of G_F. Since both G_F and $G_{F,W}$ are proper and G_F is smooth, the claim follows. ☐

Using this Lemma, we can prove the proposition:

Proof of Proposition. By blowing up if necessary, we may assume $Z := \overline{X} - X$ is a reduced effective Cartier divisor on $\overline{X}$. For a large enough embedding $\overline{X} \subset \mathbf{P}^N$, there exists a codimension $(d-1)$-plane $H \subset \mathbf{P}^N$ such that

(1) $H \cap \mathrm{Sing}(\overline{X}) = \emptyset$.
(2) $H \cap \overline{X}$ is a smooth curve on $\overline{X}$ containing F_X.
(3) $H \cap Z$ is a finite reduced subscheme.

Indeed, consider the space G_F of all codimension $(d-1)$-planes H that contain F_X for a suitably large projective embedding. Then the subspace of G_F spanned by those planes that satisfy each of the conditions above separately is a non-empty Zariski open of G_F: by the Lemma for (1) since $\dim(\mathrm{Sing}(\overline{X})) \leq d-2$ by normality, by Bertini for (2), and by the Lemma and Bertini for (3) (as Z is generically smooth, H can be chosen to avoid $\mathrm{Sing}(Z)$ by the Lemma and be transverse to Z at its intersection points by Bertini). Hence, their intersection is also a non-empty Zariski open, so there exists an H satisfying all three conditions. The construction of [dJ96, §4.11] carried out using this H to construct the projection (see the last paragraph of *loc. cit.*) then gives the desired π and f. ☐

Set $F_Y = \pi(F_X) \subset Y$ to denote the corresponding subscheme of Y[1]. Note that the properties ensured by the Proposition are preserved if we replace $(X, \overline{X})$ by $(U, \overline{X})$ where $U \subset X \subset \overline{X} \times_{\overline{Y}} Y$ is the complement of larger divisor finite étale over Y that does not meet F_X.

[1] F_Y is currently a single reduced point, though subsequent operations will entail adding more points.

3.4. **A remark on base changes** [dJ96, §2.18,4.15]**.** Consider the map $\overline{\pi} : \overline{X} \to \overline{Y}$ at the end of §3.3. In the sequel, we will need know some stability properties of $\overline{\pi}$ under strict transforms along an alteration Y, so we record:

Proposition. *Let $\pi : \mathcal{C} \to B$ be a map of integral schemes satisfying (2) and (3) from the Proposition in §3.3. Let $f : B' \to B$ be a generically étale alteration with B' integral, and let $\pi' : \mathcal{C}' \to B'$ be the strict transform of π.*

(1) $\mathcal{C}'$ is the reduction of $\mathcal{C} \times_B B'$.
(2) $\mathcal{C}'$ is integral.
(3) The map π' also satisfies properties (2) and (3) from the Proposition in §3.3.
(4) If π is smooth at a point $b \in B(k)$, then π' is smooth along $f^{-1}(b)$.
(5) If f is étale at $b \in B(k)$, then $\mathcal{C}' \hookrightarrow \mathcal{C} \times_B B'$ is an isomorphism in a neighbourhood of the fibres over $f^{-1}(b)$.
(6) If $Z \subset \mathcal{C}$ is the support of an effective Cartier divisor with $Z \to B$ finite, then the preimage Z' of Z in $\mathcal{C}'$ is also the support of an effective Cartier divisor with $Z' \to B'$ finite. If additionally $Z \to B$ is étale over some $b \in B$, then $Z' \to B'$ is étale over $f^{-1}(b)$.

Proof. These facts are standard; we sketch proofs for completeness.

(1) Clearly π' is a proper generically smooth map. Since the smooth locus of π is dense in all the fibres, the same is true for the base change $\pi_{B'} : \mathcal{C} \times_B B' \to B'$, and hence also for $\pi' : \mathcal{C}' \to B'$. Now the smooth locus of $\pi_{B'}$ is contained in $\mathcal{C}' \subset \mathcal{C} \times_B B'$ (as this locus in $\mathcal{C} \times_B B'$ is flat over B'). The density of this locus in the fibres implies: for each geometric point $y \in B'$, the closed immersion $\mathcal{C}'_y \subset \mathcal{C}_y$ has dense image, and must thus be a (set-theoretic) equality. This implies $|\mathcal{C}'| = |\mathcal{C} \times_B B'|$, so $\mathcal{C}'$ is the reduction of $\mathcal{C} \times_B B'$.
(2) All generic points of irreducible components of $\mathcal{C}'$ must lie on the generic fibre of $\mathcal{C}' \to B'$ by definition of strict transform. Since the generic fibre of π is proper smooth and geometrically connected, the same is true for the generic fibre of $\mathcal{C}' \to B'$, which shows that $\mathcal{C}'$ has only one irreducible component.
(3) This is clear from the discussion in (1).
(4) The structure sheaf of base change $\mathcal{C} \times_B B'$ has no $\mathcal{O}_{B'}$-torsion in a neighbourhood of $f^{-1}(b)$ by smoothness, so $\mathcal{C}' \subset \mathcal{C}$ is an isomorphism over these neighbourhoods.
(5) The base change $\mathcal{C} \times_B B'$ is étale over $\mathcal{C}$ in a neighbourhood of $f^{-1}(b)$, so it is already reduced as $\mathcal{C}$ is so.
(6) Locally on $\mathcal{C}'$, by (2), any regular function cutting out Z in $\mathcal{C}$ (up to passage to reduced subschemes) also pulls back to a regular function on $\mathcal{C}'$ cutting out Z' (up to passage to reduced subschemes), so the first part is clear. It is also clear that $Z' \to B'$ is finite. For the last part, after shrinking B around b, we may assume $Z \to B$ is finite étale. By (1), the canonical map $Z' \to Z \times_B B'$ is a nil-thickening. As $Z \times_B B' \to B'$ is finite étale, the scheme $Z \times_B B'$ is reduced (as B' is so), and thus $Z' \simeq Z \times_B B'$, so $Z' \to B'$ is finite étale. ☐

From now on, we will use the above stability properties without comment.

3.5. **Finding multisections** [dJ96, §4.13]**.** Let $\pi : (X, \overline{X}) \to (Y, \overline{Y})$ be the map constructed at the end of §3.3. We want to find a multisection of $\overline{\pi}$ which intersects all fibres sufficiently, and behaves well over F_Y; recall that F_Y is currently a reduced scheme supported at a single closed point of Y.

Proposition. *There exists a divisor $\overline{H} \subset \overline{X}$ and an open neighbourhood $F_Y \subset U \subset Y$ such that*

(1) The map $\overline{\pi}|_{\overline{H}} : \overline{H} \to \overline{Y}$ is finite, and finite étale over U.
(2) The inverse image of U in $\overline{H}$ lies in $X \times_Y U$.
(3) $\overline{H} \cap F_X = \emptyset$.
(4) For any geometric point $y \in \overline{Y}(\overline{k})$ and any irreducible component $C \subset \overline{X}_y$, the set-theoretic intersection $(\overline{X}/\overline{Y})^{\mathrm{sm}} \cap C \cap H$ has size at least 3.

Proof. This follows from the proof of [dJ96, §4.13]. The only thing to check is that the map $\overline{H} \to \overline{Y}$ constructed as in *loc. cit.* is étale around F_Y and that (2) and (3) above are satisfied. To see this, take the point y in [dJ96, page 70, paragraph 2] to be the unique point of F_Y, and the set U to only contain hyperplanes $\overline{H}$ that avoid the finite set $F_X \cup (\overline{X}_y - X_y) \subset \overline{X}_y$ (in addition to the requirements of *loc. cit.*). This shows that there is a Zariski open V in the set of all hyperplanes $\overline{H}$ and an open neighbourhood $F_Y \subset U \subset Y$ such that (1), (2), (3) are satisfied, and (4) is satisfied for geometric points in U. It remains to show that hyperplanes in V satisfying (4) for *all* geometric points span a non-empty Zariski open subset. Choose a point $y' \in \overline{Y} - U$. Repeating the above argument for y' shows that a non-empty Zariski open subset of hyperplanes in V satisfy (4) for over an open subset of $\overline{Y}$ strictly larger than U. Proceeding this way, we conclude by noetherian induction (see [dJ96, page 70, paragraph 4]). ☐

The map $g : (U, \overline{Y}) \to (Y, \overline{Y})$ is finite étale neighbourhood of F_Y. After replacing $\pi : (X, \overline{X}) \to (Y, \overline{Y})$ by its base change along g (i.e., shrinking X to $X \times_Y U$), we may assume $U = Y$ in the above Proposition without destroying any property ensured by the Proposition in §3.3. Let $H \subset \overline{H}$ be the inverse image of Y; note that H is smooth by (1) above. Then the induced map $\epsilon' : (H, \overline{H}) \to (Y, \overline{Y})$ of pairs is finite étale.

3.6. **Adding sections** [dJ96, §4.14-4.16]**.** Let $\pi : (X, \overline{X}) \to (Y, \overline{Y})$ be as in §3.3. Using the map ϵ' from §3.5, we construct an étale base change of π that has many sections.

Proposition. *After replacing π with its normalised base change along a suitable finite étale neighbourhood of F_Y, we can ensure:*

- *(1) π satisfies all properties of the Proposition in §3.3 except (6).*
- *(2) There exist sections $\sigma_1, \dots, \sigma_n : \overline{Y} \to \overline{X}$ such that*
  - *(a) For any geometric point $y \in \overline{Y}(\overline{k})$ and any irreducible component C of $\overline{X}_y$, the set-theoretic intersection $\cup_i \sigma_i(\overline{Y}) \cap C \cap (\overline{X}/\overline{Y})^{\mathrm{sm}}$ has size at least 3.*
  - *(b) The induced sections $\sigma_{1,Y}, \dots, \sigma_{n,Y} : Y \to \overline{X} \times_{\overline{Y}} Y$ factor through X, are pairwise distinct in each fibre over Y, and miss F_X completely.*

Proof. Consider the map $\epsilon' : (H, \overline{H}) \to (Y, \overline{Y})$ from §3.5. The map $\epsilon' : H \to Y$ is a finite étale cover. Let Y' denote the Galois closure of this map, and let $\overline{Y'}$ be the normalisation of $\overline{Y}$ in the induced map $Y' \to H \to Y$. Then we obtain a finite étale neighbourhood $\epsilon : (Y', \overline{Y'}) \to (Y, \overline{Y})$ of F_Y that is generically on Y a Galois cover. The base change of π along this neighbourhood then does the job: (1) is clear, while the construction of §3.5 shows (2). ☐

3.7. **Birth of stable curves** [dJ96, §4.17-4.22]**.** Let $(\pi, \sigma_1, \dots, \sigma_n)$ be as in §3.6. Note that the Proposition in §3.6 show that $(\overline{\pi}_Y, \sigma_{1,Y}, \dots, \sigma_{n,Y})$ is a stable curve over Y. The next step is to replace this data by a stable curve over all of $\overline{Y}$ by passing to a finite étale neighbourhood of F_Y.

Proposition. *After replacing π with its base change along a finite étale neighbourhood of F_Y, we may assume:*

- *(1) π satisfies all properties of the Proposition in §3.6 except possibly the normality of $\overline{X}$.*
- *(2) The map $\overline{\pi} : \overline{X} \to \overline{Y}$ is flat.*
- *(3) There exists a stable curve $(\mathcal{C} \to \overline{Y}, \tau_1, \dots, \tau_n) \in \mathcal{M}_{g,n}(\overline{Y})$.*
- *(4) There is a $\overline{Y}$-map $\beta : \mathcal{C} \to \overline{X}$ carrying τ_i to σ_i that restricts to an isomorphism over Y.*

Proof. The strict tranform of $\overline{\pi}$ along a suitable Y-admissible modification $\overline{Y'} \to \overline{Y}$ is flat by [dJ96, §2.19]; this property persists under base changes and hence further strict transformations. Hence, we may assume π satisfies (1) and (2). The argument in [dJ96, §4.17] constructs a finite étale map $\epsilon : (Y', \overline{Y'}) \to (Y, \overline{Y})$ such that the base change of the stable curve $(\overline{\pi}|_Y, \sigma_{1,Y}, \dots, \sigma_{2,Y})$ to Y' admits an extension to a stable curve $(\mathcal{C} \to \overline{Y}, \tau_1, \dots, \tau_n)$ over $\overline{Y'}$. Replacing π by its base change along ϵ, we may assume that (1), (2), (3) are satisfied, and also that one has an isomorphism $\beta|_Y$ as in (4). The extension of $\beta|_Y$ to a morphism over $\overline{Y}$ is de Jong's "three point lemma" proven in [dJ96, §4.18-4.22]; this is where property (2) from the Proposition in §3.6 matters. ☐

Since the $\overline{Y}$-map $\mathcal{C} \to \overline{X}$ is an isomorphism over Y, the map $(\beta^{-1}(X), \mathcal{C}) \to (X, \overline{X})$ is a finite étale neighbourhood of F_X. Thus, after replacing $(X, \overline{X})$ by this neighbourhood, we may assume that there exists a map $\pi : (X, \overline{X}) \to (Y, \overline{Y})$ and maps $\sigma_1, \dots, \sigma_n : \overline{Y} \to \overline{X}$ such that:

1. π satisfies all properties of the Proposition in §3.3 except possibly (6) and the normality of $\overline{X}$.
2. The data $(\overline{\pi} : \overline{X} \to \overline{Y}, \sigma_1, \dots, \sigma_n : \overline{Y} \to \overline{X})$ defines a stable curve over $\overline{Y}$.
3. The sections $\sigma_1, \dots, \sigma_n$, when restricted to $Y \subset \overline{Y}$, are contained in X and disjoint from F_X.

These properties are preserved by replacing π with its base change along a finite étale neighbourhood of F_Y.

3.8. **End of proof** [dJ96, §4.22-4.28]**.** Consider the data presented at the end of §3.7. After replacing π with its base change along a semistable finite étale neighbourhood of F_Y in $(Y, \overline{Y})$ (provided by induction), we may additionally assume that $(Y, \overline{Y})$ is a semistable pair. In particular, $\overline{\pi} : \overline{X} \to \overline{Y}$ satisfies all the constraints of [dJ96, §4.23]. We leave it to the reader to check that the explicit blowups constructed in [dJ96, §4.23-4.28] provide a finite étale neighbourhood $(X', \overline{X'}) \to (X, \overline{X})$ of F_X with $(X', \overline{X'})$ semistable: all steps in *loc. cit.* only entail blowing up along $\mathrm{Sing}(\overline{X})$, and hence are isomorphisms over X.

4. THE MIXED CHARACTERISTIC SETTING

Fix a perfect characteristic p field, and let K be a finite extension of $W(k)[1/p]$.

Definition 4.1. We define the category $\mathcal{P}_K$ of *pairs* as follows. The objects are (scheme-theoretically) dense open immersions $j : U \hookrightarrow \overline{U}$ where U is a smooth quasi-projective K-variety, and $\overline{U}$ is a proper flat reduced $\mathcal{O}_K$-scheme; a map is given by the obvious commutative square. We may abusively write objects of $\mathcal{P}_K$ as $(U, \overline{U})$. For a map $g : (U, \overline{U}) \to (V, \overline{V})$ in $\mathcal{P}_K$, we often write $g : U \to V$ and $\overline{g} : \overline{U} \to \overline{V}$ for the induced maps. Given a pair $(U, \overline{U})$ with $\overline{U}$ normal, we define the *special fibre* of $\overline{U}$ to be the closed fibre of $\overline{U} \to \mathrm{Spec}(\mathcal{O}(\overline{U}))$; note that $\mathcal{O}(\overline{U})$ is a finite product of orders in discrete valuation rings (and is normal if $\overline{U}$ is so). We say that a map $g : (U, \overline{U}) \to (V, \overline{V})$ is *étale* (resp. *finite étale*) if the map $U \to V$ is so. Given a finite subscheme $Z \subset V$, we say that an étale map $g : (U, \overline{U}) \to (V, \overline{V})$ is a *finite étale neighbourhood* of Z if the map $U \to V$ is finite étale in a neighbourhood of Z. Let $\mathcal{P}_K^{\mathrm{ss}} \subset \mathcal{P}_K$ be the full subcategory of *semistable pairs*, i.e., pairs $(U, \overline{U})$ where (a) $\overline{U}$ is regular, (b) the structure map $\overline{U} \to \mathrm{Spec}(\mathcal{O}(\overline{U}))$ has geometrically connected and reduced fibres, and (c) $\overline{U} - U$ is a strict normal crossings divisor. A family $\{f_i : (U_i, \overline{U_i}) \to (V, \overline{V})\}$ of maps in $\mathcal{P}_K$ or $\mathcal{P}_K^{\mathrm{ss}}$ is called an *étale cover* if each f_i is étale, and $\cup_i f_i(U_i) = U$.

Remark 4.2. Given a pair $(U, \overline{U}) \in \mathcal{P}_K$, all generic points of $\overline{U}$ lie in U by the scheme-theoretic density. If moreover $\overline{U}$ is normal, then $\pi_0(\overline{U}) = \pi_0(U)$; this is the case for $(U, \overline{U}) \in \mathcal{P}_K^{\mathrm{ss}}$. Moreover, an étale map $f : (U, \overline{U}) \to (V, \overline{V})$ of pairs is precisely a proper surjective map $\overline{U} \to \overline{V}$ which carries U to V and is étale over U.

Example 4.3. For any finite extension L/K, one has a pair $\mathrm{Spec}(L, \mathcal{O}_L) := (\mathrm{Spec}(L), \mathrm{Spec}(\mathcal{O}_L)) \in \mathcal{P}_K^{\mathrm{ss}}$. If $R \subset \mathcal{O}_L$ is a non-maximal order in L, then the pair $(\mathrm{Spec}(L), \mathrm{Spec}(R)) \in \mathcal{P}_K$ is *not* semistable. The pair $\mathrm{Spec}(K, \mathcal{O}_K)$ is a final object of $\mathcal{P}_K$, and the structure map $\mathrm{Spec}(L, \mathcal{O}_L) \to \mathrm{Spec}(K, \mathcal{O}_K)$ is étale. Another example of an étale map in $\mathcal{P}_K$ is given by normalisation: given $(U, \overline{U}) \in \mathcal{P}_K$, one obtains $(U, \overline{U}^{\mathrm{norm}}) \to (U, \overline{U})$ where $\overline{U}^{\mathrm{norm}}$ denotes the normalisation of $\overline{U}$ in U; this map is a finite étale neighbourhood of any finite subscheme of U. More generally, any U-admissible modification of $\overline{U}$, i.e., a proper birational map $\overline{V} \to \overline{U}$ which is an isomorphism over U, provides such an example.

Remark 4.4. The category $\mathcal{P}_K$ admits certain limits. For example, if $f : (X, \overline{X}) \to (Y, \overline{Y})$ is any map in $\mathcal{P}_K$ and $\epsilon : (Y', \overline{Y'}) \to (Y, \overline{Y})$ is an étale map, then the base change of f along ϵ exists, is étale over $(X, \overline{X})$, and is given by $(X', \overline{X'})$ where $X' = X \times_Y Y'$ and $\overline{X'}$ is the scheme-theoretic closure of X' in $\overline{X} \times_{\overline{Y}} \overline{Y'}$. Given a finite subscheme $F_X \subset X$, if ϵ is a finite étale neighbourhood of $f(F_X)$, then $(X', \overline{X'}) \to (X, \overline{X})$ is a finite étale neighbourhood of F_X. The induced map $(X', \overline{X'}^{\mathrm{norm}}) \to (Y', \overline{Y'})$ is called the *normalised* base change of π along ϵ. Note that $\mathcal{P}_K^{\mathrm{ss}} \subset \mathcal{P}_K$ is *not* closed under the limits just described: the regularity of $\overline{U}$ for a semistable pair $(U, \overline{U})$ can be lost when making a ramified base change.

Our main theorem is the following:

Theorem 4.5. *Let $(U, \overline{U}) \in \mathcal{P}_K$, and let F_X be a finite reduced subscheme of U. Then there exists a finite étale neighbourhood $(V, \overline{V}) \to (U, \overline{U})$ of F_X with $(V, \overline{V})$ semistable.*

As in equicharacteristic, this implies:

Theorem 4.6. *Restriction induces equivalences of topoi:* $\mathrm{Shv}_{\text{ét}}(\mathcal{P}_K) \simeq \mathrm{Shv}_{\text{ét}}(\mathcal{P}_K^{\mathrm{ss}}) \simeq \mathrm{Shv}_{\text{ét}}(\mathrm{Sm}_K)$.

5. PROOF OF THEOREM 4.5

The proof of Theorem 4.5 is similar to Theorem 2.5. To make arguments flow better, we make define:

Definition 5.1. A map $\pi : (X, \overline{X}) \to (Y, \overline{Y})$ is called a *compactified elementary fibration* if

1. $\overline{\pi} : \overline{X} \to \overline{Y}$ is a projective map with all fibres geometrically connected and equidimension 1.
2. The smooth locus of $\overline{\pi}$ is dense in all fibres.
3. $\overline{\pi}|_Y : \overline{X}|_Y \to Y$ is smooth.
4. The composition $X \hookrightarrow \overline{X}|_Y \to Y$ is an elementary fibration in the sense of Artin, i.e., $X \to Y$ is smooth, and $D := \overline{X}|_Y - X$ is a Cartier divisor in $\overline{X}|_Y$ with $D \to Y$ finite étale and of constant degree.

Maps of such fibrations are defined by the evident commutative square. We also name the following properties:

(1) π has *many sections* if there are sections $\sigma_1, \ldots, \sigma_n : \overline{Y} \to \overline{X}$ distinct in fibres over Y such that $\cup_i \sigma_i(\overline{Y})$ intersects (set-theoretically) every irreducible component of every fibre of $\overline{\pi}$ in at least 3 points.
(2) π is *stable n-pointed* if it comes equipped with sections $\sigma_1, ..., \sigma_n : \overline{Y} \to \overline{X}$ such that the datum $(\overline{\pi} : \overline{X} \to \overline{Y}, \sigma_1, ..., \sigma_n)$ defines stable n-pointed curve over $\overline{Y}$ in the sense of Deligne-Mumford.
(3) π is called *split* if for every geometric point $y \in \overline{Y}$ and every singular point $x \in \overline{X}_y$ of the fibre, there exists a section $\sigma : \overline{Y} \to \overline{X}$ with $\sigma(y) = x$; this notion will only be used in for stable π, and the sections σ going through the nodes are not part of the data, and are independent of the sections making π stable (as the latter cannot go through the nodes).

Remark 5.2. The class of compactified elementary fibrations is stable under étale base changes in $\mathcal{P}_K$. Moreover, the properties defined above are also preserved under such base changes.

For the rest of this section, fix a pair $(X, \overline{X}) \in \mathcal{P}_K$.

5.1. **Notation.** Let $d = \dim(X) = \dim(\overline{X}) - 1$. We always use Z for $\overline{X}[1/p] - X$, viewed as a closed subscheme of $\overline{X}[1/p]$ via the reduced structure. We use a subscript 0 and η to indicate passage to the special and generic fibres for $\mathcal{O}_K$-schemes.

5.2. **Preliminary reductions.** As in §3.2, we may assume $d > 0$ and that $\overline{X}$ is normal and projective. Moreover, we may also assume $k = \overline{k}$ by Artin approximation; alternatively, one may keep expanding K to a slightly larger unramified extension as necessary in the proof. By [dJ96, Lemma 2.13], we may assume that $\overline{X}$ is normal with a geometrically reduced special fibre. By extension of K and passage to connected components, we may assume that $\mathcal{O}(\overline{X}) = \mathcal{O}_K$. In particular, $\overline{X}$ is normal, and the geometric fibres of $\overline{X} \to \mathrm{Spec}(\mathcal{O}_K)$ are connected and reduced.

5.3. **A presentation result.**

Proposition. *There exists a finite morphism $f : \overline{X} \to \mathbf{P}^d_{\mathcal{O}_K}$, a point $p \in \mathbf{P}^d(\mathcal{O}_K)$, and an $\mathcal{O}_K$-flat line ℓ going through p such that:*

(1) f is étale over an open in $\mathbf{P}^d_{\mathcal{O}_K}$ that contains p.
(2) $f^{-1}(\ell)_\eta$ is a smooth curve on $\overline{X}^{\mathrm{sm}}_\eta$ containing F_X and meeting Z transversally in a finite reduced subscheme.

Proof. The idea of the proof is to simply solve the problem separately over the generic and special fibres, and then observe that the space of solutions in either case is large to come admit a common integral point. In more detail, choose a large enough projective embedding $X \hookrightarrow \mathbf{P}^N_{\mathcal{O}_K}$, . By Bertini, there exists an $\mathcal{O}_K$-flat $(N - d + 1)$-plane $L \subset \mathbf{P}^N_{\mathcal{O}_K}$ such that

(1) $L_0 \cap \overline{X}_0$ is a generically smooth curve on $\overline{X}_0$.
(2) $L_\eta \cap \overline{X}_\eta$ is a smooth curve on $\overline{X}_\eta$ containing F_X.
(3) $L_\eta \cap \mathrm{Sing}(\overline{X}_\eta) = \emptyset$.
(4) $L_\eta \cap Z$ is a finite reduced subscheme of Z.

Indeed, let G denote the scheme of all $(N - d + 1)$-planes in $\mathbf{P}^N_{\mathcal{O}_K}$. Then G is a proper smooth $\mathcal{O}_K$-scheme, and G_0 is the space of all $(N - d + 1)$-planes in $\mathbf{P}^N_k$. By Bertini, the planes L_0 with $L_0 \cap \overline{X}_0$ generically smooth span a non-empty Zariski open subset $V_0 \subset G_0$. Similarly, the planes L_η satisfying the analogs of (2), (3) and (4) span a non-empty Zariski open dense subset $V_\eta \subset G_\eta$. Choosing an $L \in G(\mathcal{O}_K)$ specialising to a point of V_0 and V_η (possible by smoothness of G) then gives the desired L. By (1), we can choose $\mathcal{O}_K$-flat linear subspaces $W' \subset W \subset L$ such that

(a) W' has relative dimension $(N - d - 1)$ over $\mathcal{O}_K$ and $W' \cap \overline{X} = \emptyset$.
(b) W has relative dimension $(N - d)$ over $\mathcal{O}_K$, $W_0 \cap \overline{X}_0$ is a finite reduced subscheme of $(L_0 \cap \overline{X}_0)^{\mathrm{sm}} \cap \overline{X}_0^{\mathrm{sm}}$, and the finite reduced subscheme $W_\eta \cap \overline{X}_\eta$ of $L_\eta \cap \overline{X}_\eta$ does not meet F_X.

By (a), projecting away from W' gives a finite morphism $f : \overline{X} \to \mathbf{P}^d_{\mathcal{O}_K}$. The plane W defines defines a point $p \in \mathbf{P}^d(\mathcal{O}_K)$ with f finite étale at p by (b). The plane L defines a line ℓ through p in $\mathbf{P}^d_{\mathcal{O}_K}$ with $f^{-1}(\ell) = L \cap \overline{X}$. In particular, $f^{-1}(\ell)_\eta$ is a smooth curve in $\overline{X}^{\mathrm{sm}}_\eta$ by (2) and (3) that meets Z transversally by (4). □

Remark 5.3. With some extra work, one can also arrange for the line ℓ found above to satisfy: the projective closure $\overline{F_X} \subset \overline{X}$ is contained in the smooth locus of $f^{-1}(\ell) \to \mathrm{Spec}(\mathcal{O}_K)$. The key is to first apply Neron's dilation arguments and perform some X-admissible blowups on $\overline{X}$ to assume: $\overline{F_X}$ lies in the smooth locus of $\overline{X} \to \mathrm{Spec}(\mathcal{O}_K)$.

5.4. **Finding good projections.**

Lemma. *Let $\pi : W \to Y$ be a projective smooth map whose fibres are geometrically connected curves. Let $X \subset W$ be an open subscheme. Fix a finite reduced closed subscheme $F_Y \subset Y$, and let $F_X \subset X \times_Y F_Y$ be a finite reduced subscheme of the fibres. Then there exists an open $F_Y \subset U \subset Y$ and an open $F_X \subset V \subset \pi^{-1}(U) \subset X$ such that $W_U - V$ is finite étale of constant degree over U.*

Proof. We give a proof when $F_Y = \{y\}$ is a point, leaving the generalisation to the reader. After shrinking Y around y, we may assume Y is affine. Choose a π-relatively very ample line bundle $\mathcal{O}(1) \in \mathrm{Pic}(W)$ and an integer $n > 0$ such that $H^1(W, I_{W_y}(n)) = 0$. Choose a section $s_y \in H^0(W_y, \mathcal{O}(n)|_{W_y})$ such that $Z(s_y) \subset W_y$ is a finite reduced subscheme that contains $F_X \cup (W_y - X_y)$; this is possible after replacing n by a larger integer. Choose a lift $s \in H^0(W, \mathcal{O}(n))$ of s_y; this is possible as $H^1(W, I_{W_y}(n)) = 0$. Let $Z = Z(s) \subset W$ be the zero-locus; by using Y-flat hypersurfaces $\mathbf{P}(H^0(W, \mathcal{O}(n)))$ that do not contain any fibre of π, we may choose s so that $Z \to Y$ is flat. After shrinking Y further if necessary, by properness of Z and $W - X$ relative to Y and semicontinuity, we may assume that $W - X \subset Z$. Set $V = W - Z$. Then $F_X \subset V \subset X$ by construction. Moreover, $Z = W - V$ is finite flat over Y, and unramified at y. By shrinking Y further, we may assume that $Z \to Y$ is finite étale of constant degree. In particular, V provides the desired open. $\square$

Lemma. *Fix an integer $d > 1$, a closed point p in $\mathbf{P}^d_k$, a line L through p, and an open subset $U \subset \mathbf{P}^d_k$ containing p. Then there exists some integer $n \geq 3$ and a finite set $S = \{H_1, \ldots, H_n\}$ of hyperplanes in $\mathbf{P}^d_k$ such that*

1. *$p \notin H_i$ for any i.*
2. *$H_i \cap L$ is a closed point z_i in U for all $i \in \{1, \ldots, n\}$.*
3. *The points z_i from (2) are pairwise distinct, i.e., $z_i \neq z_j$ for $i \neq j$.*
4. *For any line L' going through p, there exist three distinct indices $i_1, i_2, i_3 \in \{1, \ldots, n\}$ such that*
   (a) *$H_{i_j} \cap L'$ is a closed point w_j in U for all $j \in \{1, 2, 3\}$.*
   (b) *The points w_1, w_2, and w_3 are all distinct.*

Proof. Let $\mathbf{P}$ be the space of all lines through p, so $\mathbf{P} \simeq \mathbf{P}^{d-1}_k$. Let G be the space all hyperplanes in $\mathbf{P}^d_k$ that miss p, so G is the complement of a hyperplane in the dual $\mathbf{P}^d_k$. Choose H_1, H_2, and H_3 in G that satisfy (1), (2), and (3); set $\{z_i\} = H_i \cap L$. Then the set $S = \{H_1, H_2, H_3\}$ satisfies (4) for the line L, and hence for all lines L' in a non-empty Zariski open $V \subset \mathbf{P}$. Our strategy is to add more hyperplanes to S so that (1), (2), and (3) are still satisfied, and (4) is satisfied for a strictly larger open subset than V. Set $Z = S \cap L$, i.e., $Z = \{z_1, z_2, z_3\}$ where $z_i = H_i \cap L$. Now choose a line $L' \in \mathbf{P} - V$. Pick 3 distinct points $z_4, z_5, z_6 \in (L \cap U) - Z$ and 3 distinct points $w_4, w_5, w_6 \in (L' \cap U) - (S \cap L')$. Choose hyperplanes $H_4, H_5, H_6 \in G$ with H_i joining z_i with w_i for $i \in \{4, 5, 6\}$. Replacing S with $S \cup \{H_4, H_5, H_6\}$ gives a finite set S that still satisfies (1), (2) and (3). Moreover, (4) is now true for an open subset $V' \subset \mathbf{P}$ that is strictly larger than V. Proceeding this way, we conclude by noetherian induction. $\square$

Corollary. *After replacing $(X, \overline{X})$ by a finite étale neighbourhood of F_X, we may assume that there exist:*

1. *A compactified elementary fibration $\pi : (X, \overline{X}) \to (Y, \overline{Y})$.*
2. *A bundle $\pi' : \mathbf{P} \to \overline{Y}$ of projective lines.*
3. *A finite $\overline{Y}$-morphism $f : \overline{X} \to \mathbf{P}$.*
4. *A finite set $\{\sigma_1, \ldots, \sigma_n : \overline{Y} \to \mathbf{P}\}$ of sections of π'.*

These can be chosen to satisfy:

(a) *The étale locus $V \subset \mathbf{P}$ of f is dense every fibre of π'.*
(b) *The induced sections $\sigma_{1,Y}, \ldots, \sigma_{n,Y} : Y \to \overline{\mathbf{P}}|_Y$ are distinct and factor through $V|_Y$.*
(c) *The scheme $\cup_i \sigma_i(\overline{Y})$ meets every fibre of π' (set-theoretically) in at least 3 distinct points lying in V.*

Proof. Choose a finite morphism $f' : \overline{X} \to \mathbf{P}^d_{\mathcal{O}_K}$ with a point $p \in \mathbf{P}^d(\mathcal{O}_K)$ and an $\mathcal{O}_K$-flat line L through p satisfying the conclusions of the Proposition in §5.3. Applying the above lemma with this choice of p and the open set U being the étale locus of f' gives a finite set $S = \{H_1, \ldots, H_n\}$ of $\mathcal{O}_K$-flat hyperplanes satisfying the conclusions of the Lemma. Since p does not meet F_X, by shrinking, we may replace $(X, \overline{X})$ by $(X, \mathrm{Bl}_{f'^{-1}(p)}(\overline{X}))$. Set $\mathbf{P} = \mathrm{Bl}_p(\mathbf{P}^d_{\mathcal{O}_K})$ and $\overline{Y} = \mathbf{P}(T_p) \simeq \mathbf{P}^{d-1}_{\mathcal{O}_K}$. This gives us a finite morphism $f : \overline{X} \to \mathbf{P}$ and a projective line bundle $\pi' : \mathbf{P} \to \overline{Y}$, which gives (2) and (3). Set $\overline{\pi} : \overline{X} \to \overline{Y}$ to be the composite. Since f' is étale at p, the blown-up map f is étale along the exceptional divisor E of $\mathbf{P}$. In particular, the étale locus of f meets every fibre since E is a section of π', which gives (a). Since the hyperplanes in S miss p, they inverse images under the blowup map give sections $\sigma_1, \ldots, \sigma_n : \overline{Y} \to \mathbf{P}$

of π', which gives (4). The line L defines an $\mathcal{O}_K$-point $q \in \overline{Y}$ with $\overline{\pi}^{-1}(q)$ being the preimage in $\overline{X}$ of the strict transform of L. In particular, the generic fibre of $\overline{\pi}^{-1}(q)$ is a smooth curve on $\overline{X}_\eta^{\mathrm{sm}}$ and meets $\overline{X} - X$ transversally in a finite reduced set. By construction of S, there exists a sufficiently small open neighbourhood Y of $q_\eta \in \overline{Y}_\eta$ such that the sections σ_i are all distinct over Y and the induced map $\sigma_{i,Y} : Y \to \mathbf{P}|_Y$ factors through $V|_Y$; this gives (b). The properties on S also ensure that $(\cup_i H_i) \cap L' \cap U$ has size at least 3 (set-theoretically) for any $\mathcal{O}_K$-flat line L' through p. Taking strict transforms shows (c). It remains to prove (1), i.e., to shrink X to a Zariski open around F_X such that the induced map $(X, \overline{X}) \to (Y, \overline{Y})$ is a compactified elementary fibration. The fibres have equidimension 1 as they are connected subschemes of $\overline{X}$ equipped with a finite map to $\mathbf{P}^1$. The smooth locus is dense in all fibres by (a), and the rest can be proven using the Lemma above. □

Using π, we define the finite reduced subscheme $F_Y := \pi(F_X) \subset Y$.

5.5. **Adding sections.** The next step is to ensure an abundance of good sections:

Proposition. *After replacing π with its base change along a finite étale neighbourhood of F_Y in $(Y, \overline{Y})$, we may assume that π has many sections.*

Proof. Consider the data presented in the Corollary in §5.4. Set $\overline{H} \subset \overline{X}$ to be the preimage of $\overline{H'} := \cup_i \sigma_i(\overline{Y}) \subset P$ under f. Note that each $\sigma_i(\overline{Y})$ is an effective Cartier divisor on $\mathbf{P}$ since π' is a smooth relative curve. By the stability of Cartier divisors under union and pullbacks under dominant morphisms of integral schemes, we see that $\overline{H'}$ and $\overline{H}$ are both Cartier divisors. By construction, $\overline{H'} \to \overline{Y}$ is finite étale over Y, and that $\overline{H'}|_Y \to Y$ factors through V. Since f is étale over V, it follows that $\overline{H} \to \overline{Y}$ is finite étale over Y. Moreover, for any irreducible component C of a fibre of $\overline{\pi}$ over $y \in \overline{Y}$, the set-theoretic intersection $\overline{H} \cap C$ maps onto $\overline{H'} \cap \mathbf{P}_y$. By assumption on the sections σ_i, it follows that $\overline{H} \cap C$ has at least 3 distinct points. The desired claim follows by taking Galois closures as in [dJ96, Lemma 5.6] and the following two remarks: (a) the flatness of $\overline{H} \to \overline{Y}$ can be ensured after replacing π with its base change along a suitable map $(Y, \overline{Y'}) \to (Y, \overline{Y})$ with $\overline{Y'} \to \overline{Y}$ a Y-admissible modification that flattens $\overline{H} \to \overline{Y}$, and (b) instead of choosing ad hoc Galois superextensions as in *loc. cit.*, we use the Galois closure to ensure good behaviour over Y. □

Write $\sigma_1, ..., \sigma_n : \overline{Y} \to \overline{X}$ be the sections resulting from the above proposition. By construction, these are distinct in the fibers over Y, and there are at least 3 going through each fiber. In particular, the $\overline{X}|_Y \to Y$ equipped with the sections $\sigma_1|_Y, ..., \sigma_n|_Y : Y \to \overline{X}|_Y$ defines an object of $\mathcal{M}_{g,n}(Y)$, i.e., a smooth n-pointed stable curve over Y.

5.6. **Stabilisation.** We now replace π with a stable curve fibration.

Proposition. *After replacing $(X, \overline{X})$ with a finite étale neighbourhood of F_X, we may assume that there exists a stable n-pointed compactified elementary fibration $\pi : (X, \overline{X}) \to (Y, \overline{Y})$ with sections $\tau_1, ..., \tau_n$ to $\overline{\pi}$.*

Proof. This step is exactly analogous to §3.7, so we only summarise the idea. First, using moduli of stable curves with level structure, one shows that $(\overline{X}|_Y \to Y, \sigma_1, \dots, \sigma_n) \in \mathcal{M}_{g,n}(Y)$ extends to a stable n-pointed curve $(\mathcal{C} \to \overline{Y}, \tau_1, \dots, \tau_n) \in \overline{\mathcal{M}}_{g,n}(\overline{Y})$ after replacing $(Y, \overline{Y})$ by a finite étale cover. Moreover, the 3-point lemma shows that there is a $\overline{Y}$-map $\beta : \mathcal{C} \to \overline{X}$ extending the isomorphism $\mathcal{C}|_Y \to \overline{X}|_Y$ and carrying τ_i to σ_i. Replacing $(X, \overline{X})$ with $(\beta^{-1}(X), \mathcal{C})$ and the σ_i with the τ_i's then gives the claim. □

5.7. **Splitting stable curves.** Next, we must find sections going through the singular points in the fibres of $(X, \overline{X}) \to (Y, \overline{Y})$. For this, we use the following abstract statement:

Lemma. *Let $f : \mathcal{C} \to S$ be a semistable curve over a quasi-projective excellent integral scheme S. Fix finitely many points $s_1, \dots, s_n \in S$ such that f is smooth over each s_i, as well as dense open subschemes $U_i \subset \mathcal{C}_{s_i}$. Then there exists a Cartier divisor $H \subset \mathcal{C}$ such that*

(1) $H \cap \mathcal{C}_{s_i}$ is a finite reduced subscheme contained in U_i.
(2) The map $H \to S$ is finite, and finite étale in a neighbourhood of each s_i.
(3) H contains $\mathrm{Sing}(f)$.

Proof. Fix an ample line bundle on $\mathcal{C}$, so all Serre twists are taken with respect to this bundle. Let $I_{\mathrm{Sing}(f)}$ be the ideal sheaf $\mathrm{Sing}(f)$, and let $I_{\mathrm{Sing}(f) \cap \mathcal{C}_s} \subset \mathcal{O}_{\mathcal{C}_s}$ denote the ideal sheaf of $\mathrm{Sing}(f) \cap \mathcal{C}_s \subset \mathcal{C}_s$ for any $s \in S$. For each $s \in S$, the natural map $I_{\mathrm{Sing}(f)} \to I_{\mathrm{Sing}(f) \cap \mathcal{C}_s}$ is surjective. Moreover, one can choose an integer $n \gg 0$ such that $H^0(\mathcal{C}, I_{\mathrm{Sing}(f)}(n)) \to H^0(\mathcal{C}_s, I_{\mathrm{Sing}(f) \cap \mathcal{C}_s}(n))$ is also surjective; this is possible by semicontinuity. Fix some such n, and consider the corresponding family $\mathbf{P} = \mathbf{P}(H^0(\mathcal{C}, I_{\mathrm{Sing}(f)}(n))) \subset \mathbf{P}(H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(n)))$ of hypersurfaces in $\mathcal{C}$; thus, each $H \in \mathbf{P}$ is given by a hypersurface $H \subset \mathcal{C}$ that contains $\mathrm{Sing}(f)$. Using the afore-mentioned surjectivity, together with the fact that $\mathrm{Sing}(f)$ does not contain any component of any fibre of f, one can show: for each $s \in S$, there is some open neighbourhood $s \in V_s \subset S$ and a dense open subset $W_s \subset \mathbf{P}$ such that each $H \in W_s$ is finite and flat over $V_s \subset S$. By compactness, there is some dense open $W \subset \mathbf{P}$ such that each $H \in W$ is finite and flat over S. It remains to show that, after shrinking W further if necessary, each $H \in W$ is finite étale over each s_i with $H \cap \mathcal{C}_{s_i} \subset U_i$. For this, observe that $I_{\mathrm{Sing}(f)} \to \mathcal{O}_C$ and $H^0(\mathcal{C}, I_{\mathrm{Sing}(f)}(n)) \to H^0(C, \mathcal{O}_C)$ are surjective, where $C = \cup_i \mathcal{C}_{s_i} \subset \mathcal{C}$ is the displayed finite union of smooth projective curves in $\mathcal{C}$; here we use that $\mathrm{Sing}(f) \cap C = 0$, so $I_{\mathrm{Sing}(f) \cap C} \subset \mathcal{O}_C$ is an equality. Using this, one argues similarly as above. □

Proposition. *After replacing $(X, \overline{X})$ by a finite étale neighbourhood of F_X, we may assume that there exists a split stable n-pointed compactified elementary fibration $\pi : (X, \overline{X}) \to (Y, \overline{Y})$ with sections $\tau_1, ..., \tau_n$ to $\overline{\pi}$.*

Proof. Consider the stable compactified elementary fibration $\pi : (X, \overline{X}) \to (Y, \overline{Y})$ constructed in §5.6. As in the previous lemma, choose a Cartier divisor $\overline{H} \subset \overline{X}$ containing $\mathrm{Sing}(\overline{\pi})$ such that $\overline{H} \to \overline{Y}$ is finite and finite étale over some $U \subset Y$ containing F_Y. Replacing Y with U and setting H to be the inverse image of Y then gives a finite étale map $(H, \overline{H}) \to (Y, \overline{Y})$ in $\mathcal{P}_K$. Taking Galois closures and base changes as in §5.5 then provides the desired split curve. □

5.8. **End of proof.** Consider the split stable curve $\pi : (X, \overline{X}) \to (Y, \overline{Y})$ with sections $\tau_1, ..., \tau_n$ to $\overline{\pi}$ constructed in §5.7. By induction and base change, we may assume that $(Y, \overline{Y})$ is a semistable pair. At this point we are in the situation of [dJ96, §6.15]. Since the construction of [dJ96, §6.16] only involves blowing up in $\mathrm{Sing}(\overline{X})$, we are through.

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