A workshop on almost purity at Ann Arbor

Goal

The main goal of this workshop is to understand Faltings' almost purity theorem which, roughly speaking, can be thought of as a version of Abhyankar's lemma that is valid in mixed characteristic. In the process of reaching this goal, we will learn basics of almost ring theory (this is the main new technical ingredient introduced by Faltings to "soften" mixed characteristic rings), and also see some applications to p-adic comparison theorems conjectured by Fontaine. To achieve this goal in a short time span, we will focus on the case of good reduction.

Dates and location

The workshop will take place from May 13th through May 15th in 4096 East Hall.

Format

This is a small workshop dedicated to a single theorem. Hence, we will have roughly three talks a day for three days, with the time in between talks used to make sure that the material is well understood.

Schedule

- Friday, May 13th:
  - 9:30 - 11:00 : Bhargav Bhatt: *Introduction and Tate's theorem*
  - 1:00 - 2:30 : Karl Schwede: *Topological invariance of the almost étale site*
  - 3:00 - 4:30 : Kazuma Shimomoto: *Normalised lengths in almost ring theory*
- Saturday May 14th:
  - 9:00 - 10:30 : Mircea Mustaţă: *Almost purity in characteristic* p
  - 11:00 - 12:00 : **Special talk** by Kazuma Shimomoto: *Local Bertini theorems in mixed characteristic and applications to Iwasawa theory*
  - 2:00 - 3:30 : Ray Heitmann: *Almost purity outside dimension* 3
  - 4:00 - 5:30 : Paul Roberts: *Fontaine rings*
  - Dinner: There will be a scheduled dinner for the participants of the conference on Saturday evening; a head count will be taken on the first day.
- Sunday May 15th:
  - 9:30 - 11:00 : Remi Lodh: *Almost purity in general*
  - 1:00 - 2:30 : **Special talk** by Ruochuan Liu: *Relative* p*-adic Hodge theory*
  - 3:00 - 4:30 : Bhargav Bhatt: *Perfectoid spaces, following Scholze*

The 90 minute slots above can be only partially used! Also, there will be coffee available each morning.

List of talks

The following is a rough plan of a route we could take; if you feel that any topic deserves more (or less) attention than a single talk, please let me (bhargav.bhatt@gmail.com) know! Each talk should be ≤ 80 minutes long.

1. Basics and Tate's theorem. This talk should setup the basics of almost ring theory (see [Ols09, §2] and [GR03, §3]), state and prove Tate's theorem in the one-dimensional case (see [Tat67, §3] and [Fal88, Theorem 1.2]), and then state Faltings' theorem in general (see [Ols09, Theorem 2.17] and [Fal02, page 192]). We could also split this talk into two, with the second one dedicated to Tate's theorem and its applications in arithmetic. (Bhatt)

2. Deformation theory of almost étale maps. This talk should discuss the "topological invariance" of the almost étale site (see [GR03, Theorem 3.2.18] and [Fal02, pages 189-192]). (Schwede)

3. Normalised lengths in almost ring theory. This notion is a substitute for the usual notion of lengths that makes sense over the big rings that show up in the *statement* of almost purity. This talk should discuss the definition of normalised lengths, additivity properties, behaviour under Frobenius twisting, and what it means for normalised lengths to be almost 0 (see [GR, §3.3] and [Fal02, pages 200-204]). (Shimomoto)

4. Proof of almost purity in characteristic p. This talk should discuss the purity theorem in characteristic p (see [GR03, Theorem 3.5.28]); this proof is easier than its mixed characteristic counterpart, and yet illuminating. (Mustaţă)

5. Proof of almost purity in mixed characteristic in dimension not equal to 3. This talk should prove the purity theorem in dimension ≤ 2 (using basic commutative algebra, see [GR, Theorem 3.1.35]), and in dimension ≥ 4 (by following the characteristic p approach, see [GR, §3.3]). (Heitmann)

6. Fontaine rings and related schemes. Fontaine rings are certain mixed characteristic deformations of characteristic p algebras that admit a Frobenius lift and some additional structures; they enter in the proof of purity in dimension 3 in a decisive way. This talk should discuss the basic objects of this theory from the ring theoretic perspective (see [GR, §1.3] and [Fon94, §1.2 and §1.3]), and the associated picture involving certain formal schemes and their coherent cohomology (see [GR, §3.4]). (Roberts)

7. Proof of almost purity in all dimensions. The title is self-explanatory; the main references are [GR, Theorem 3.5.34] and [Fal02, pages 192-198]. (Lodh)

8. Applications to p-adic comparision theorems. This talk should explain what almost purity says about the p-adic comparison theorems conjectured by Fontaine. For example, one could explain [Fal88, Theorem 4.5] which gives a concrete relation between étale cohomology and de Rham cohomology for certain affines, and then briefly summarize what goes into globalising this result. By sticking to the Hodge-Tate case (which is surely enough to interest pure geometers), all mention of crystalline cohomology and B_{crys} can be avoided. (No speaker scheduled)

9. Peter Scholze's generalization. Starting a deeply ramified nonarchimedean field K with positive characteristic residue field, Scholze introduces the notion of a *perfectoid* K-algebra. When two such fields K and K' are related by Fontaine's theory (in particular, K has characteristic 0 while K' has characteristic $p > 0$), Scholze proves that the resulting categories of perfectoid algebras are equivalent by a process called *tilting* (which is very similar to constructions in Fontaine's theory). Moreover, he shows that tilting preserves the étale topology; this fact can be regarded as a generalization of almost purity. The goal of this talk should be explain what perfectoid algebras are, and describe the tilting procedure; here the key point that should be emphasized is how the *almost* vanishing of the relative cotangent complex of (integral) perfectoid algebras allows one to seamlessly move between characteristic 0 and characteristic p without ever taking Witt vectors (except of the base). (Bhatt)

In addition, there will be two special talks.

1. Kazuma Shimomoto will talk on mixed characteristic local Bertini theorems and applications to Iwasawa theory.

2. Ruochuan Liu will talk about his work with Kedlaya on relative p-adic Hodge theory

Comment on references

Note that all references to [GR] above are to version 2 of their preprint, which is not the newest version available. The reason for this is psychological: version 2 contains a proof in the case of good reduction and is 170 pages long, while the newest version contains a much more general statement and is close to 800 pages long.

Participants (not including graduate students)

- Raymond Heitmann (Texas)
- Mel Hochster (Michigan)
- Ruochuan Liu (Michigan)
- Remi Lodh (Utah)
- Mircea Mustaţă (Michigan)
- Paul Roberts (Utah)
- Karl Schwede (PSU)
- Kazuma Shimomoto (Meiji)
- Anurag Singh (Utah)
- Karen Smith (Michigan)
- Kevin Tucker (Utah and Princeton)
- Wenliang Zhang (Michigan)
- Bhargav Bhatt (Michigan)

References

[Fal88] Gerd Faltings. p-adic Hodge theory. *J. Amer. Math. Soc.*, 1(1):255–299, 1988.

[Fal02] Gerd Faltings. Almost étale extensions. *Astérisque*, (279):185–270, 2002. Cohomologies p-adiques et applications arithmétiques, II.

[Fon94] Jean-Marc Fontaine. Le corps des périodes p-adiques. *Astérisque*, (223):59–111, 1994. With an appendix by Pierre Colmez, Périodes p-adiques (Bures-sur-Yvette, 1988).

[GR] Ofer Gabber and Lorenze Ramero. Foundations of p-adic hodge theory (version 2). Available at `http://arxiv.org/pdf/math/0409584v2`.

[GR03] Ofer Gabber and Lorenzo Ramero. *Almost ring theory*, volume 1800 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2003.

[Ols09] Martin C. Olsson. On Faltings' method of almost étale extensions. In *Algebraic geometry—Seattle 2005. Part 2*, volume 80 of *Proc. Sympos. Pure Math.*, pages 811–936. Amer. Math. Soc., Providence, RI, 2009.

[Tat67] J. T. Tate. $p-divisible\ groups.$. In *Proc. Conf. Local Fields (Driebergen, 1966)*, pages 158–183. Springer, Berlin, 1967.