The relative FL theory and A_{inf} representations w Frobenius

we will consider this complex

- $A\Omega_{\mathfrak{X}}$ $A\Omega_R$ with coefficients?

- k perfect field of char $p>0$, $W=W(k)$ $K=\mathrm{Frac}\,W$, $\overline{K}$ alg closure, $C=\widehat{\overline{K}}$.

  $\mathcal{O}_{\overline{K}}\subset\overline{K}$, $\mathcal{O}_C\subset C$ rings of integers

$\mathrm{Spf}(R)\longrightarrow\mathrm{Spf}(W)$ smooth, connected

s.t. $\exists\ t_1,\dots,t_d\in R^\times$ coordinates

$\mathbb{K}=\mathrm{Frac}\,R$, $\overline{\mathbb{K}}$

$\overline{R}$ = the int. closure of R in the maximal unramified extension of

$R[1/p]$ in $\overline{\mathbb{K}}$.

$G_R:=\mathrm{Aut}(\overline{R}/R)\supset\Delta_R:=\mathrm{Aut}(\overline{R}/R\mathcal{O}_{\overline{K}})$.

Consider $(M,\mathrm{Fil}^r M,\nabla,\Phi)$, where

- M finite free R-module
- $\mathrm{Fil}^r M\subset M$ decreasing filtration st. $\mathrm{Fil}^0 M=M$, $\mathrm{Fil}^{p-1}M=0$

$\exists\ e_1,\dots,e_N\in M$, basis $r_1,\dots,r_N\in[0,p-2]$

s.t. $\mathrm{Fil}^r M=\bigoplus_{r_i\geq r}R\,e_i$

$\nabla: M\longrightarrow M\otimes_R\Omega^1_R$ top quasi-nilpotent, integrable connection.

s.t. $\nabla(\mathrm{Fil}^r M)\subset\mathrm{Fil}^{r-1}M\otimes\Omega_R$.

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- $\varphi : R \to R$ lifting of Frob

$\phi : (\varphi^* M, \varphi^* \nabla) \to (M, \nabla)$ horizontal, R-linear, where

$\varphi^*(\ ) = (\ ) \otimes_{R,\varphi} R$,

s.t.

- $\Phi(\varphi^*(\mathrm{Fil}^r M)) \subset p^r M$ — natural filtration is: $\sum_{0 \le s \le r} p^{r-s} \varphi^*(\mathrm{Fil}^s M)$
- $\sum_{0 \le r \le p-2} p^{-r} \phi(\varphi^*(\mathrm{Fil}^r M)) = M$

The category of these tuples is $MF^{\nabla}_{[0,p-2],\mathrm{free}}(R, \phi)$

$$\exists\ T_{\mathrm{crys}} : MF^{\nabla}_{[0,p-2],\mathrm{free}}(R, \phi) \longrightarrow \mathrm{Rep}_{\mathbb{Z}_p,\mathrm{free}}(G_R)$$

fully faithful

G_R, φ $\curvearrowright$ $A_{\inf}(\overline{R}) = W(\varprojlim_{\mathrm{Frob}} \overline{R}/p\overline{R})$

$\cap$

$A_{\mathrm{crys}}(\overline{R})$

$\varepsilon := [(\zeta_{p^n} \bmod p)]$ $\quad \mu = [\varepsilon] - 1 \in A_{\inf}(\mathcal{O}_{\overline{K}}) =: A_{\inf}$.

$\cap$ $\varprojlim_{\mathrm{Frob}} \overline{R}/p\overline{R}$

$$\xi = (\varphi^{-1}(\mu)/\mu)^{-1} \quad \tilde{\xi} = \varphi(\mu)/\mu \ (= [p]_q \text{ from Scholze's 2nd lecture})$$

$\mathcal{M}_{[0,p-2],\text{free}}(A_{\inf}(\overline{R}), G_R, \varphi)$

Objects : $\mathcal{M}$ finite free $A_{\inf}(\overline{R})$-modules

+ semilinear continuous action of G_R

+ G_R-equivariant semilinear map

$$\varphi : \mathcal{M} \longrightarrow \mathcal{M}.$$

s.t. $\exists\, e_1, \dots, e_N \in \mathcal{M}$ basis $r_1, \dots, r_N \in \mathbb{N} \cap [0, p-2]$:

st. $\xi^{-r_i}\varphi(e_i)$ form a basis of $\mathcal{M}$.

Theorem 1 (T):

(1). $\exists$ canonical fully faithful functor

$$MF^{\nabla}_{[0,p-2],\text{free}}(R, \Phi) \to \mathcal{M}_{[0,p-2],\text{free}}(A_{\inf}(\overline{R}), G_R, \phi)$$

$$\begin{array}{ccc} \Psi & T_{A_{\inf}} & \Psi \\ M & \longrightarrow & \mathbb{M} \end{array}$$

$$T = T_{crys}(M) \in \mathrm{Rep}_{\mathbb{Z}_p}(G_R)$$

(2). $T^* = \mathrm{Hom}_{A_{\inf}(\overline{R}), \varphi}(\mathbb{M}, A_{\inf}(\overline{R}))$

G_R-equivariant.

(3) (2) induces

$$\mathbb{M} \xhookrightarrow[\text{inj}]{} T \otimes_{\mathbb{Z}_p} A_{\inf}(\overline{R}) \subset \mu^{-(p-2)}\mathbb{M}.$$

(4) $\mathbb{M}/\mu\mathbb{M} = (A_{\inf}(\overline{R})/\mu)^N$ as G_R-representation

(5) consider another M', $\mathbb{M}'$, T':

$$\begin{array}{ccc} f:\ T\otimes A_{\inf}[1/\mu] & \longrightarrow & T'\otimes A_{\inf}[1/\mu] \\ \cup & & \cup \\ \mathbb{M} & \underset{\exists}{\dashrightarrow} & \mathbb{M}' \end{array}$$

G_R-equiv, compatible w φ.

Remark: • (5), (1) $\Rightarrow$ T_{crys} is fully faithful.

• (5) follows from $\left(\mu^r A_{\inf}(\bar{R}) / \mu^{r+1} A_{\inf}(\bar{R})\right)^{G_R} = 0$ if $r \neq 0$.

Construction of $TA_{\inf}$:

$$MF^{\nabla}_{[0,p-2],\mathrm{free}}(R,\phi) \longrightarrow MF_{[0,p-2],\mathrm{free}}(A_{crys}(\bar{R}),\phi,G_R)$$

is fully faithful.

$\rightsquigarrow$ "evaluation on Spec $A_{crys}(\bar{R})/p^n$" $\bar{R}/p^n$.

$$\begin{array}{c} MF_{[0,p-2],\mathrm{free}}(A_{crys}(\bar{R}),\phi,G_R) \\ \uparrow \ \cong \text{ equiv (Fontaine-Wach)} \\ M_{[0,p-2],\mathrm{free}}(A_{\inf}(\bar{R}),\phi,G_R) \end{array}$$

$$A\Omega_R(M) \overset{\text{def}}{=} L\eta_\mu R\Gamma(\Delta_R, TA_{\inf}(M))$$

$DR(M) := M \otimes_R \Omega^{\bullet}_R$.

Theorem 2 (T): (1) $p \geq 5$; $\exists$ can isom $A\Omega_R(M) \overset{\mathbb{L}}{\otimes}_{A_{\inf}} A_{\inf}/\mu \simeq A_{\inf}/\mu \,\hat{\otimes}_W DR(M)$.

compatible w G_R-action, φ.

$\rightsquigarrow$

—— $\overset{\mathbb{L}}{\otimes} O_C \simeq O_C \hat{\otimes}_W DR(M)$

—— $\overset{\mathbb{L}}{\otimes} W \simeq DR(M)$.

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(2) $\exists$ canonical map

$$A_{crys} \hat{\otimes}_W DR(M) \to A_{crys} \hat{\otimes}_{A_{inf}} A\Omega(M)$$

computation. $A_{inf}^{\square}(R) = R \hat{\otimes}_W A_{inf} \circlearrowleft \Gamma, \varphi.$

$$\begin{array}{l} A_{inf}^{\square}(R) \\ \downarrow \quad \longleftarrow \quad T_i \longmapsto [T_i^{b}] \\ A_{inf}(R_\infty) \end{array}$$

$$R_\infty = R[t_i^{p^{-\infty}}]$$

$$\gamma_1, \dots, \gamma_d \in \Gamma = \mathrm{Aut}(R_\infty / R) \simeq \mathbb{Z}_p^d$$

$\exists$ descent $TA_{inf}^{\square}(M)$ of $TA_{inf}(M)$.

$\circlearrowleft \Gamma, \varphi.$

$$A\Omega_R(M) = K^{\bullet}\left(TA_{inf}^{\square}(M) : \frac{\gamma_1 - 1}{\mu}, \dots \frac{\gamma_d - 1}{\mu}\right)$$

Koszul complex

$$A_{crys} \hat{\otimes}_{A_{inf}} TA_{inf}^{\square}(M) \simeq M \hat{\otimes}_W A_{crys}$$

$$\gamma_i(m \otimes 1) = \sum_{n \in \mathbb{N}} \nabla\left(\frac{\partial}{\partial T_i}\right)^n (m) \otimes \mu^{[n]}$$