jt work in progress w B. Bhatt, M. Morrow

THH vs cryst. coh. and $A\Omega$

1st part : crystalline cohomology

2nd part : $A\Omega$

Everything in this talk is p-complete.

cyclotomic spectrum

$$\operatorname{cone}\left(X_{hC_p} \xrightarrow{Nm} X^{hC_p}\right)$$
$$\|$$
X spectrum $\circlearrowleft S^1$ + $\varphi_p : X \longrightarrow X^{tC_p}$

$S^1 \simeq S^1/C_p$-equivariant

$$TC(X) := eq\left(X^{hS^1} \underset{can}{\overset{\varphi_p^{hS^1}}{\rightrightarrows}} (X^{tC_p})^{h(S^1/C_p)}\right)$$
$$(X^{tC_p})^{h(S^1/C_p)} = \operatorname{cone}\left(Nm\,(X_{hC_p})^{hS^1/C_p} \longrightarrow (X^{hC_p})^{hS^1/C_p}\right), \quad (X^{hC_p})^{hS^1/C_p} = X^{hS^1}$$

$= \operatorname{map}_{CycSp}(\mathbb{S}, X)$

Lemma (~Tate Orbit Lemma): If X is bounded below,

$$(X^{tC_p})^{hS^1/C_p} \xleftarrow{\sim} X^{tS^1} = \operatorname{cone}\left(Nm\, \Sigma X_{hS^1} \to X^{hS^1}\right)$$

If R is an $\mathbb{E}_\infty$-ring spectrum, $TC(R) = TC(THH(R))$.

$$\left(THH(R)^{hS^1}, THH(R)^{tS^1}, \varphi_p \cdots\right)$$

Goal: Make these things explicit for smooth k-algebra (k perfect)

smooth $\mathcal{O}_C$-algebra ($C/\mathbb{Q}_p$ alg. closed complete)

Base: $\pi_* THH(k) = k[u] \quad |u| = 2$

$\Downarrow$ Bokstedt

$$\pi_* THH(k)^{hS^1} = W(k)[u,v]/uv-p \qquad |u| = 2,\ |v| = -2$$

$$E_2 : H^*(S^1, \pi_* THH(k)) \Rightarrow \pi_* THH(k)^{hS^1}$$

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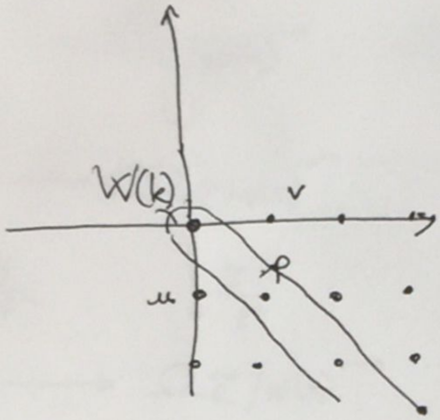


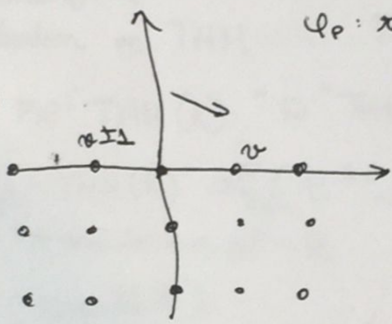
k in all even degrees.

can: $u \longmapsto p v^{-1}$, $v \longmapsto v$

$$\pi_* THH(k)^{tS^1} = W(k)[v^{\pm 1}]$$

$$\Uparrow$$

$$E_2^{**} = \hat{H}^*(S^1, \pi_* THH(k))$$



$$\varphi_p : \pi_* THH(k)^{hS^1} \longrightarrow \pi_* THH(k)^{tS^1}$$

$$W(k)[u,v]/(uv-p) \qquad W(k)[v^{\pm 1}]$$

$u \longmapsto v^{-1}$

$v \longmapsto v.$

<u>Remark</u>: These rings appear in Fontaine–Jansen theory of gauges.

Now let R be a smooth k-algebra. Want to find the following structures in THH:

$$\varphi \circlearrowright R\Gamma_{crys}(R/W(k)) \simeq W\Omega_{R/k}.$$

$$\cup|$$

$$R\Gamma_{crys}(R/W(k)) \xleftarrow{\ \frac{\varphi}{p^i}\ } \mathrm{Fil}^i R\Gamma_{crys}(R/W(k)) \quad \text{Nygaard filtration}$$

$$\mathrm{gr}^i R\Gamma_{crys}(R/W(k)) \simeq \tau^{\le i} \Omega^{\bullet}_{R/k}. \quad \underset{\text{Cartier}}{H^j} = \left(\Omega^j_{R/k}\right)^{(p)} \quad j \le i$$

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If $\tilde{R}/W(k)$ smooth lift, $\tilde{\varphi}$ lift of φ.

$$
\begin{array}{lccccccccc}
R\Gamma_{crys}: & 0 \to & \tilde{R} & \xrightarrow{\nabla} & \Omega^1_{\tilde{R}/W(k)} & \to \cdots & \Omega^d_{\tilde{R}/W(k)} & \to 0 \\
 & & \cup & & \cup & & & \\
\mathrm{Fil}^i & 0 \to & p^i\tilde{R} & \longrightarrow & p^{i-1}\Omega^1_{\tilde{R}/W(k)} & \to \cdots & p^{\max(i-d,0)}\Omega^d_{\tilde{R}/W(k)} & \to 0 \\
 & & \downarrow \frac{\tilde{\varphi}}{p^i} & & \downarrow \frac{\tilde{\varphi}}{p^i} & & \downarrow \frac{\tilde{\varphi}}{p^i} & \\
 & 0 \to & \tilde{R} & \longrightarrow & \Omega_{\tilde{R}/W(k)} & \to \cdots & \Omega^d_{\tilde{R}/W(k)} & \to 0
\end{array}
$$

$$H^j(gr^i) = p^{i-j}\Omega^j_{\tilde{R}} / p^{i-j+1}\Omega^j_{\tilde{R}} = \Omega^j_{R/k}.$$

<u>Step 1</u>: Define descending, sep. filtration on THH $\qquad \mathrm{Fil}^0 = \mathrm{THH}$.

$$\mathrm{Fil}^i\,\mathrm{THH}(R) \text{ "}\subseteq\text{" } \mathrm{THH}(R)$$

s.t. $gr^i\,\mathrm{THH}(R) \simeq F_{R*}(\tau^{\leq i}\Omega^\bullet_{R/k})[2i]$. $\quad i \geq 0$.

$\hookrightarrow$ module over $gr^0 = R$

Recall: <u>Thm</u> (Hesselholt)

R smooth $/k$ $\quad \pi_*\mathrm{THH}(R)$

$$
\begin{array}{c|cccc}
 & gr^0 & \xrightarrow{B} gr^1 & \xrightarrow{B} gr^2 & \\
0 & R & & & \\
1 & & \Omega^1_{R/k} & & \\
2 & & Ru & \Omega^2_{R/k} & \\
3 & & & \Omega^1_{R/k}u & \Omega^3_{R/k} \\
4 & & & Ru^2 \quad \Omega^2_{R/k}u & \Omega^4_{R/k}
\end{array}
$$

Want "this" filtration.

<u>Construction</u>: use <u>Thm</u> (Bhatt) + ε : THH satisfies fpqc descent. (for normal rings)

. try to define filtration fpqc locally.

more carefully (pro)-syntomic locally.

Want to descend along $R \to R_{perf}$ (pro)-syntomic

$$THH(R) \simeq \lim \left(THH(R_{perf}) \rightrightarrows THH(R_{perf} \otimes_R R_{perf}) \rightrightarrows \cdots \right)$$

$R_{perf} \otimes_R R_{perf}$ ($\otimes \ldots$) "regular semiperfect"
Zariski locally the quotient of a perfect ring by regular sequences.

(e.g. $R = \mathbb{F}_p[T]$
$\mathbb{F}_p[T_1^{1/p^\infty}, T_2^{1/p^\infty}] / (T_1 - T_2)$)

<u>Lemma</u>: If S is regular semiperfect,
$\mathbb{L}_{S/\mathbb{F}_p}$ is fin. proj. module in deg 1.
$\Rightarrow$ $\Lambda^i \mathbb{L}_{S/\mathbb{F}_p}$ are in degree i.

If S is regular semiperfect

0. $\cdots$ S $\quad gr^0$
1.
2. $S \cdot u \quad \mathbb{L}_{S/\mathbb{F}_p}[-1] \quad gr^1$
3.
4. $S u^2 \quad \mathbb{L}_{S/\mathbb{F}_p}[-1] \cdot u \quad (\Lambda^2 \mathbb{L}_{S/\mathbb{F}_p})[-2] \quad gr^2$

<u>Define</u>: $Fil^i THH(S) = \tau_{\geq 2i} THH(S)$; for smooth R by (pro-)syntomic descent.

<u>Thm</u> (BMS) R smooth$/k$ $\quad gr^i THH(R) \simeq \left(\tau^{\leq i} \Omega^\bullet_{R/k} \right)[2i]$

as desired.

<u>Crystalline cohomology</u> <u>Thm</u> (BMS) The same (pro-) syntomic descent ~~works~~ to define filtrations on $THH(R)^{hS^1} \xrightarrow[can]{\varphi} THH(R)^{tS^1}$.

$$gr^0 THH(R)^{hS^1} \overset{can}{=} gr^0 THH(R)^{tS^1} = R\Gamma_{crys}(R/W(k))$$

φ acting on $R\Gamma_{crys}$; $gr^0 THH(R)^{hS^1} \xrightarrow{\varphi} gr^0 THH(R)^{tS^1}$

$$gr^i \underbrace{THH(R)^{tS^1}}_{} = R\Gamma_{crys}(R/W(k))\, v^{-i}$$

$i \leq 0$ $\Big\|$ alg over $THH(k)^{tS^1}$

$\approx$ 2-periodic $v^{\pm 1}$

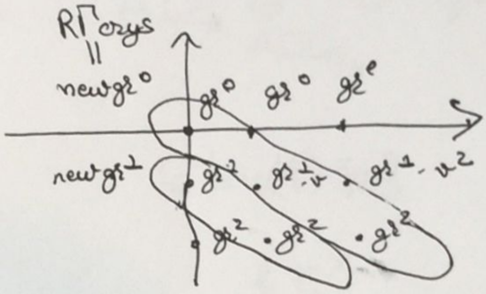
$gr^i THH(R)^{hS^1}$

Nygaard

$$\begin{array}{ccc} gr^i THH(R)^{hS^1} & = & Fil^i R\Gamma_{crys}(R/W(k))[2i]. \\ \big\downarrow \varphi_{THH} & & \big\downarrow \\ gr^i THH(R)^{tS^1} & \simeq & R\Gamma_{crys}(R/W(k))[2i] \end{array}$$

$i \geq 0$

$THH(R)^{hS^1}$



__2nd part__: $A\Omega$

structures you (want to) get. Let R be a smooth $\mathcal{O}_C$-algebra. Fix an etale map

$$\mathcal{O}_C[T_1^{\pm 1}, \dots, T_d^{\pm 1}] \longrightarrow R$$

$\rightsquigarrow \exists!$ etale deformation $\quad A_{inf}[T_1^{\pm 1}, \dots, T_d^{\pm 1}] \longrightarrow \tilde{R}$ (complete for top on A_{inf})

$\circlearrowleft \varphi \qquad \circlearrowleft \varphi$

$$A\Omega_R: \; 0 \to \tilde{R} \xrightarrow{\nabla_q} \Omega^1_{\tilde{R}/A_{inf}} \xrightarrow{\nabla_q} \cdots \xrightarrow{\nabla_q} \Omega^d_{\tilde{R}/A_{inf}} \to 0.$$

$q = [\varepsilon] \in A_{inf}$.

$$\nabla_q(T_i^n) = \underbrace{[n]_q}_{\frac{q^n-1}{q-1}} \cdot T_i^{n-1}\, dT_i.$$

$$Fil^i A\Omega: \; 0 \to [p]_q^i \tilde{R} \longrightarrow [p]_q^{i-1} \Omega^1_{\tilde{R}/A_{inf}} \to \cdots \to [p]_q^{\max(i-d,0)} \Omega^d_{\tilde{R}/A_{inf}} \to 0$$

P. Scholze

$$\begin{tikzcd}
0 \arrow[r] & {[p]_q^i \tilde{R}} \arrow[r] \arrow[d, "\frac{\varphi}{[p]_q^i}"] & {[p]_q^{i-1}\Omega^1_{\tilde{R}/A_{inf}}} \arrow[r] & \cdots \arrow[r] & {[p]\,\Omega^d_{\tilde{R}/A_{inf}}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tilde{R} \arrow[r, "\nabla_{q^p}"'] & \Omega^1_{\tilde{R}/A_{inf}} \arrow[r] & \cdots \arrow[r] & \Omega^d_{\tilde{R}/A_{inf}} \arrow[r] & 0
\end{tikzcd}$$

<u>Note</u>: $\tilde{\varphi}\left(\frac{dT_i}{T_i}\right) = \frac{dT_i^p}{T_i^p} = [p]_q \cdot T_i^{p-1}\frac{dT_i}{T_i^p} = [p]_q \frac{dT_i}{T_i}$

Everything with THH works the same using "regular semiperfectoid" rings.

Question about descent to $\mathcal{O}_K$ | R smooth / $\mathcal{O}_K$, $K/\mathbb{Q}_p$ fin.

$A\Omega_R \otimes_{A\Omega_{\mathcal{O}_K}} \mathfrak{S}$ gives a descent to $\mathfrak{S} \subseteq A_{inf}$ of $A\Omega_R \hat{\otimes}_{\mathcal{O}_K} \mathcal{O}_C$

$\mathfrak{S} = W(k)[[u]]$.

$p \neq 2$

$$A\Omega_{\mathbb{Z}_p} \simeq A\Omega_{\mathbb{Z}_p[\zeta_p]}^{hF_p^\times}$$

$$A\Omega_{\mathbb{Z}_p[\zeta_p]} = \left(\begin{array}{ccc} \mathbb{Z}_p[[q-1]] & \longrightarrow & (q-1)\,\mathbb{Z}_p[[q-1]] \\ f(q) & \longmapsto & f(q^{1+p}) - f(q) \end{array}\right)$$

THH: functor on all $\mathbb{Z}_p$-algebras.

Endow this w (pro-) syntomic topology.

locally regular semiperfectoid: take $\tau_{\geq 2i}$-filtration

$$\mathrm{gr}^\bullet \mathrm{THH}^{tS^1} = \mathrm{gr}^\bullet \mathrm{THH}^{hS^1} = A\Omega$$

$\circlearrowleft_\varphi$ $\qquad$ $\circlearrowleft \varphi$.